2.2 The \( n \times n \) Identity Matrix

Notation: The set of \( n \times n \) matrices with real entries is denoted \( M_n(\mathbb{R}) \).

Example 2.2.1 \( A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \) and let \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Find \( AI \) and \( IA \).

Solution:

\[
AI = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2(1) + 3(0) & 2(0) + 3(1) \\ -1(1) + 2(0) & -1(0) + 2(1) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = A
\]

\[
IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1(2) + 0(-1) & 1(3) + 0(2) \\ 0(2) + 1(-1) & 0(3) + 1(2) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = A
\]

Both \( AI \) and \( IA \) are equal to \( A \): multiplying \( A \) by \( I \) (on the left or right) does not affect \( A \).

In general, if \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is any \( 2 \times 2 \) matrix, then

\[
AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A
\]

and \( IA = A \) also.

Definition 2.2.2 \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is called the \( 2 \times 2 \) identity matrix (sometimes denoted \( I_2 \)).

Remarks:

1. The matrix \( I \) behaves in \( M_2(\mathbb{R}) \) like the real number 1 behaves in \( \mathbb{R} \) - multiplying a real number \( x \) by 1 has no effect on \( x \).

2. Generally in algebra an identity element (sometimes called a neutral element) is one which has no effect with respect to a particular algebraic operation.

   For example 0 is the identity element for addition of numbers because adding zero to another number has no effect.

   Similarly 1 is the identity element for multiplication of numbers.

   \( I_2 \) is the identity element for multiplication of \( 2 \times 2 \) matrices.

3. The \( 3 \times 3 \) identity matrix is \( I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) Check that if \( A \) is any \( 3 \times 3 \) matrix then

\[
AI_3 = I_3A = A.
\]
Definition 2.2.3 For any positive integer \( n \), the \( n \times n \) identity matrix \( I_n \) is defined by

\[
I_n = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & 0 & \ddots & \vdots \\
0 & \ldots & \ldots & 1
\end{pmatrix}
\]

(\( I_n \) has 1’s along the “main diagonal” and zeroes elsewhere). The entries of \( I_n \) are given by:

\[
(I_n)_{ij} = \begin{cases}
1 & i = j \\
0 & i \neq j
\end{cases}
\]

Theorem 2.2.4

1. If \( A \) is any matrix with \( n \) rows then \( I_n A = A \).

2. If \( A \) is any matrix with \( n \) columns, then \( A I_n = A \).

(i.e. multiplying any matrix \( A \) (of admissible size) on the left or right by \( I_n \) leaves \( A \) unchanged).

Proof (of Statement 1 of the Theorem): Let \( A \) be a \( n \times p \) matrix. Then certainly the product \( I_n A \) is defined and its size is \( n \times p \).

We need to show that for \( 1 \leq i \leq n \) and \( 1 \leq j \leq p \), the entry in the \( i \)th row and \( j \)th column of the product \( I_n A \) is equal to the entry in the \( i \)th row and \( j \)th column of \( A \).

\[
\begin{pmatrix}
0 \\
\vdots \\
0 \ldots 0 1 0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
A_{1j} \\
\vdots \\
\ldots A_{ij} \\
\vdots \\
A_{nj}
\end{pmatrix}
= \begin{pmatrix}
\vdots \\
\vdots \\
\ldots \ldots \ldots
\end{pmatrix}
\]

\((I_n A)_{ij}\) comes from the \( i \)th row of \( I_n \) and the \( j \)th column of \( A \).

\[
(I_n A)_{ij} = (0)(A)_{1j} + (0)(A)_{2j} + \ldots + (1)(A)_{ij} + \ldots + (0)(A)_{nj}
\]

\[
= (1)(A)_{ij}
\]

\[
= (A)_{ij}
\]

Thus \((I_n A)_{ij} = (A)_{ij}\) for all \( i \) and \( j \) - the matrices \( I_n A \) and \( A \) have the same entries in each position. Then \( I_n A = A \).

The proof of Statement 2 is similar. \( \square \)