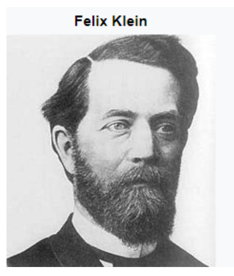
The Formulation of the Group Concept

Introduction

The evolution of group theory began in 1770 and extended to the 20th century, but the major developments occured in the 19th century from detailed study of permutations and of solutions of polynomial equations. The three main areas which gave rise to group theory are geometry, number theory and algebraic equations.

Geometry

- Though it was known that infinite groups occurred in geometry, this wasn't given much thought until the second half of the 19th century. Felix Klein suggested that, to understand geometry, it's important to study the infinite groups occurring in geometry and how they act on spaces.
- Klein published the Erlanger program in 1872. He considered geometry as a series of problems in group theory. With every geometry Klein associated an underlying group of symmetries. He was the first to propose that group theory, a branch of mathematics that used algebraic methods to abstract the idea of symmetry, was the best way of organising geometrical knowledge. His use of groups brought order to geometry.
- Using some of the basic concepts of Klein, Henri Poincaré developed the thesis that all of Euclidean geometry was actually based on the group concept. By this point groups was becoming centre stage in mathematics.



Number Theory

- In 1761, before the notion of group theory, Euler examined the remainders of powers of a number modulo n. His work is not stated in group theoretic terms but he provides an example of the decomposition of an abelian group into cosets of a subgroup. He also proves a special case of the order of a subgroup being a divisor of the order of the group.
- In 1801 Gauss studied Euler's work further and elaborated on it, giving a basis for a lot of the theory of abelian groups. Again not in group theoretic terms he proved that there is a subgroup for every number dividing the order of a cyclic group.
- He also looked at binary quadratic forms and examined the behaviour of forms under transformations and substitutions. He partitions forms into classes and then defines a composition on the classes. Gauss proves that the order of composition of three forms is immaterial, or in other words, the associative law holds.
- In fact Gauss has a finite abelian group and later (in 1869) Schering, who edited Gauss's works, found a basis for this abelian group

Algebraic Equations

- Permutations were first studied by Lagrange in 1770. His main objective was to find out why cubic and quartic equations could be solved algebraically.
- Ruffini published work in 1799 demonstrating the insolubility of the general quintic equation. Ruffini introduces groups of permutations and he divides these into cyclic groups (permutazione) and non-cyclic groups (permutazione composta). The non-cyclic groups are then divided into intransitive groups, transitive imprimitive groups and transitive primitive groups.
- In 1844, Cauchy introduces the notation of powers of permutations, with the power zero giving the identity permutation. He proves that two permutations having the same cycle structure is the same as the permutations being conjugate
- It was Abel in 1824 who gave the first accepted proof of the insolubility of the quintic.
- In 1831 Galois was the first to understand that the algebraic solution of an equation was related to the structure of a group of permutations related to the equation.
- Betti was the first to prove that Galois' group associated with an equation was in fact a group of permutations in the modern sense.
- Hölder proved the uniqueness of the factor groups in a compostion series, now called the Jordan-Hölder theorem and published the result in 1889.

Timeline

- **1761** Modular Arithmetic (*Euler*)
- **1770** Permutations (*Lagrange*)
- **1799** Claim that quintic equations could not be solved algebraically (*Ruffini*)
- **1805** Development of theory of permutations (*Cauchy*)
- **1824** Proof of insolubility of the quintic (*Abel*)
- **1831** Discovery that algebraic solutions of equations are related to groups of permutations (*Galois*)
- **1844** Notation of powers of permutations introduced (*Cauchy*)
- **1854** Abstract Groups (*Cayley*)
- **1870** Permutation Groups (*Jordan*)
- **1872** Erlanger Program (*Klein*)
- **1873** Lie Groups (*Lie*)
- **1889** Proof of uniqueness of factor groups in a composition series (*Hölder*)

Modern Day Definition of a Group

A group G is a non-empty set equipped with a binary operation *, such that:

- * is an associative operation. This means that for any elements x, y, z of G (x * y) * z = x * (y * z).
- Some element id of G is an identity element for *. This means that for every element x of G id * x = x * id = x.
- For every element x of G there is an element x^{-1} of G that is an inverse of x with respect to *.

This definition is now a starting point in the study of group theory, although it was not the starting point in the development of the subject. It arose from several decades of work by numerous mathematicians.

References

- *The Development of Group Theory* available at MacTutor History of Mathematics at the University of St Andrews website.
- MA3343 Lecture Notes
- *History of Group Theory* Wikipedia

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