The Group of Symmetries of the Tetrahedron

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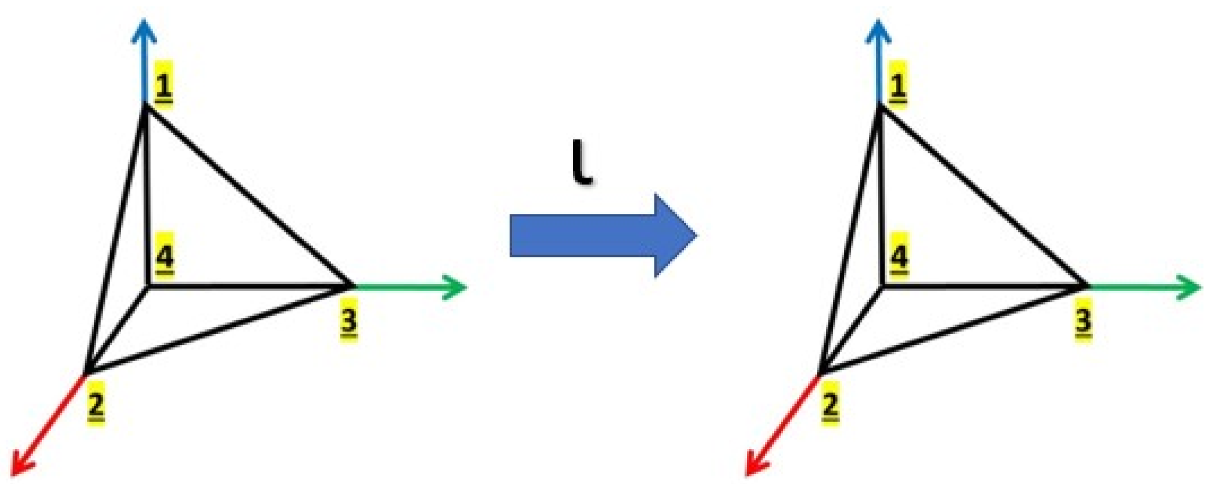
Introduction

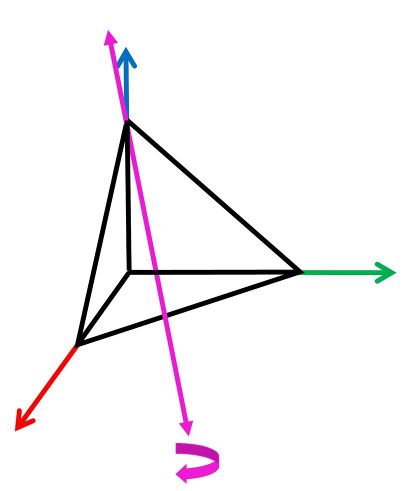
- A regular tetrahedron is a four-sided polygon, having equilateral triangles as faces.
- There are 24 symmetries of a regular tetrahedron, comprised of 12 rotational and 12 reflectional symmetries. In this project we are going to focus specifically on its rotational symmetries.

Identity Element

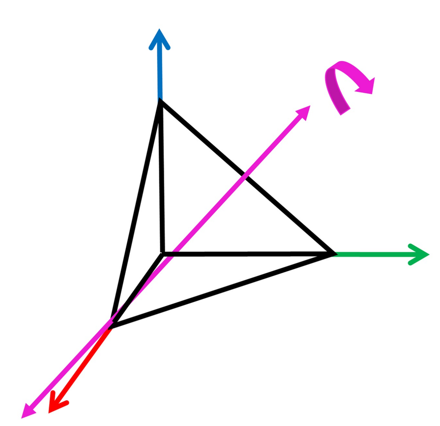


Rotations about a Single Axis

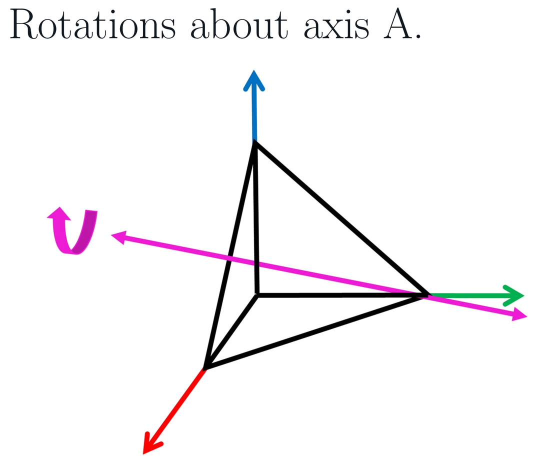
- There are 4 axes of rotational symmetry, denoted here as A, B, C, and D.
- The rotational axes pass through each vertex and the corresponding midpoint of the opposite face.
- To obtain the first eight elements, one of the four vertices is held fixed and the tetrahedron is rotated by 120 degrees, and then 240 degrees.
- The **identity element** is the result of a single rotation about any axis of 360 degrees.



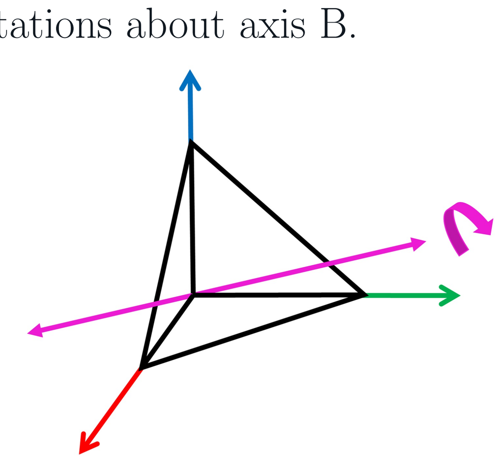
Rotations about axis A.



Rotations about axis B.

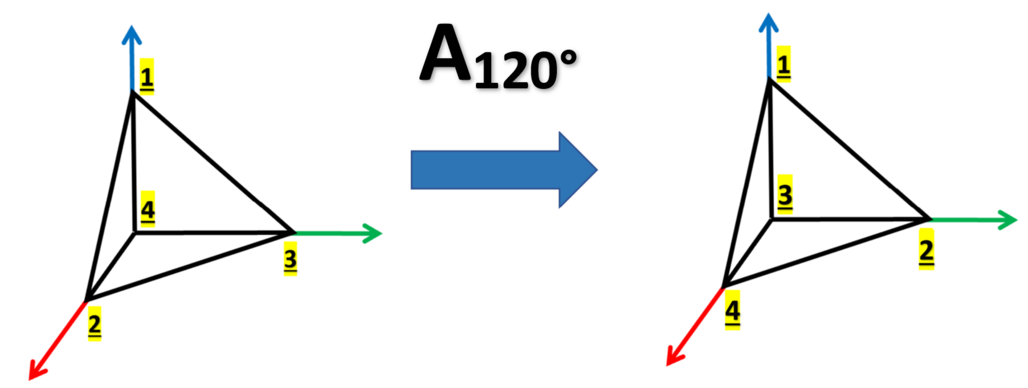


Rotations about axis C.



Rotations about axis D.

Rotations as Permutations



- The image above depicts the single anti-clockwise rotation through 120 degrees about the axis A. Written as a permutation this is $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$.
- Similarly, the rotation through 240 degrees is $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$.
- This follows for the rotations about the axes B, C and D, resulting in a total of **8 distinct elements**.

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$\epsilon = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

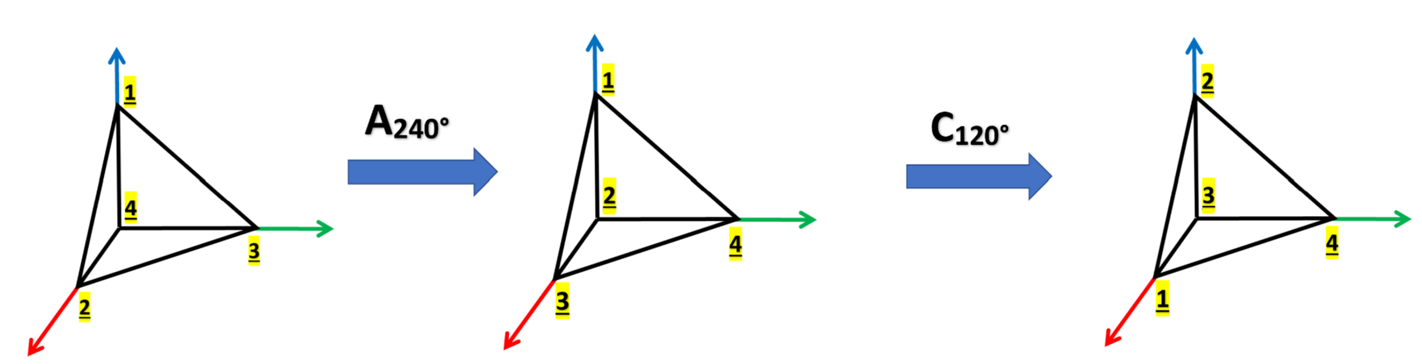
$$\zeta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$\eta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

Rotations about a Combination of Axes

- There are **three distinct elements** produced by a composition of a rotation of 240 degrees through a single axis, and a rotation of 120 degrees through a second, different axis.
- For example consider the composition of a rotation through axis A, and a rotation through axis C.



- This composition of rotations can be written as $\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$.
- By the same method we also have the two distinct elements:

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Multiplication Table

	ι	α	β	γ	δ	ε	ζ	η	θ	ρ	λ	σ
ι	ι	α	β	γ	δ	ε	ζ	η	θ	ρ	λ	σ
α	α	β	ι	ζ	ρ	θ	σ	δ	λ	η	ε	γ
β	β	ι	α	σ	η	λ	γ	ρ	ε	δ	θ	ζ
γ	γ	θ	ρ	δ	ι	β	λ	ζ	σ	ε	η	α
δ	δ	σ	ε	ι	γ	ρ	η	λ	α	β	ζ	θ
ε	ε	δ	σ	θ	λ	ζ	ι	β	ρ	γ	α	η
ζ	ζ	λ	η	ρ	α	ι	ε	σ	γ	θ	δ	β
η	η	ζ	λ	β	σ	δ	ρ	θ	ι	α	γ	ε
θ	θ	ρ	γ	λ	ε	σ	α	ι	η	ζ	β	δ
ρ	ρ	γ	θ	α	ζ	η	δ	ε	β	ι	σ	λ
λ	λ	η	ζ	ε	θ	γ	β	α	δ	σ	ι	ρ
σ	σ	ε	δ	η	β	α	θ	γ	ζ	λ	ρ	ι

An element in the left-hand column is performed first, followed by an element in the top row.

Isomorphic to S_4

- The symmetric group S_4 is the group of all permutations of length 4. It is of order 4!.
- We have shown that 12 distinct permutations of length 4 are equivalent to the 12 distinct rotations of the tetrahedron.
- Similarly, this can be shown for the 12 distinct orientation-reversing reflections of the tetrahedron.
- If these two subgroups of order 12 are joined to form a subgroup of 24 distinct permutations of length 4, we can now compare them to the 24 elements of S_4, and see that this subgroup must be all of S_4.
- Thus we have demonstrated the isomorphic property of the group of symmetries of the tetrahedron to S_4.

References

Pesek, P. (1966).
The Group of Symmetries of a Regular Tetrahedron. [online] Ojs.library.okstate.edu.
Available at: https://ojs.library.okstate.edu/osu/index.php/OAS/article/download/4476/
[Accessed 25 Nov. 2019].