Generating Sets of Symmetric Groups

Generating Sets

- A generating set of a group is a subset such that every element of the group can be expressed as a combination (under the group operation) of finitely many elements of the subset and their inverses.
- If G = $\langle S \rangle$, then we say that S generates G, and the elements in S are called generators or group generators.

Symmetric Groups

- The group consisting of all permutations of a set of n elements is called the symmetric group of degree n and denoted S_n.
- S_n has order n!, (the number of permutations of n objects)

Objective of this poster

- With this poster we hope to show how S_n the symmetric group of degree n, can be generated by its transpositions.
- A transposition is a permutation which exchanges two elements and keeps all others fixed. (it is a cycle of length 2)

Pre-Reading

- Cycles or Cycle Permutations are subsets of a permutation whose elements trade places with each other. Cycles can generate S_n.
- For example, in the permutation group (4 2 1 3), (1 4 3) is a 3-cycle and (2) is a 1-cycle. The permutation group (4 2 1 3) is a subset of S_4

Proof

- For n ≥ 2, S_n is generated by it's transpositions This is clear for n = 1 and n = 2. For n ≥ 3, we note (1) = $(12)^2$ and every cycle of length > 2 is a product of transpositions:

$$(i_1 i_2 \dots i_k) = (i_1 i_2)(i_2 i_3) \dots (i_{k-1} i_k). \tag{1}$$

For example

$$(13526) = (13)(35)(52)(26). \tag{2}$$

Since the cycles generate S_n, and products of transpositions give us all cycles, the transpositions generate S_n.

Quick examples of Symmetric groups

- S_3 is all the permutations of 3 objects; A B C for example
- A B C
- B C A
- A C B
- B A C
- C A B
- C B A
- We can clearly see that there is 6 permutations of 3 objects. Which is cleary 3!.

S_3 Generated by its Transpositions

- To begin we have (A B C)

A B C
(Transposing B and C)
A C B
(Transposing A and B)
B C A
(Transposing A and C)
B A C
(Transposing C and B)
C A B
(Transposing A and B)
C B A

- Clearly by just swapping two elements at a time we can generate all of S_3. Therefore S_3 is generated by the product of transpositions.

To Conclude

- Since cycles generate S_n and cycles can be written as the product of transpositions, therefore, S_n can be generated by transpositions.

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