Cayley Graphs

Chloe Conlon 17337976 Grace Hanlon 17475456 Kym Murray 17414662

Introduction

Cayley graphs are graphs that are associated to a group and a set of generators for that group.

Cayley Graphs

- Arthur Cayley was an English mathematician. Cayley made an important contribution to the algebraic theory of curves and surfaces, group theory, linear algebra, combinatorics and elliptic functions.
- Two groups are said to be *isomorphic* to each other if they become identical after the relabelling of their elements.
- Given a group G and a generating set X, then every element in G is assigned to a vertex in X, such that there are directed edges going from

$$a \in G \rightarrow ax \in G$$

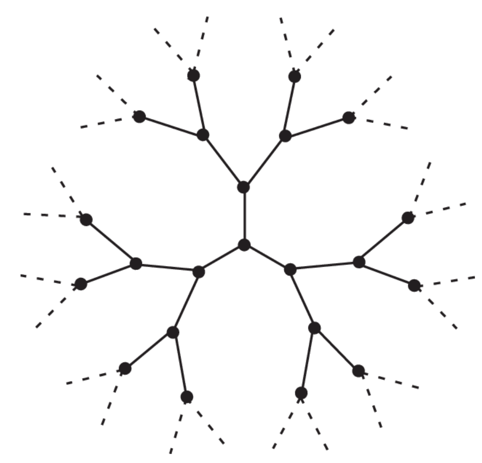
given a colour assigned to X

Bethe Lattice

Introduced by the German physicist in 1935, the Bethe Lattice is an infinite, cycle-free graph, with each node connected to z neighbours, where z is the coordination number.

It is a rooted tree with all nodes arranged in shells around the origin of the lattice.

The Bethe Lattice is equivalent to the Cayley graph of a free group on n generators.



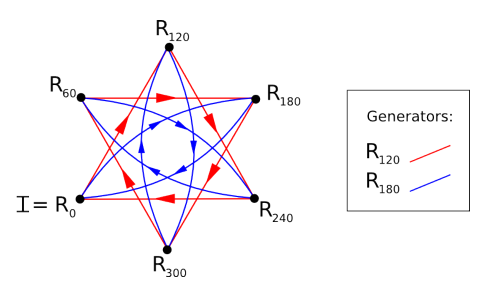
Construction of a Cayley Graph

Once we find a group that is generated by some finite collection elements, we can construct a directed graph. Thus, every group element corresponds to an isometry.

- Here is the construction of a Cayley graph for a group G with generators $a_1, a_2, ..., a_n$ in 3 steps:
- Draw one vector for every group element.
- For every generator a_j, connect vertex g to ga_j by a directed edge from g to ga_j. Label the edge with the generator.
- Repeat step 2 for every element (i.e. vertex) $g \in G$.

Example Z_6

Draw the Cayley graph for Z_6, with just one generator, namely the 60 degree rotation. However, we can also generate it with two generators: rotations by 120 and 180.

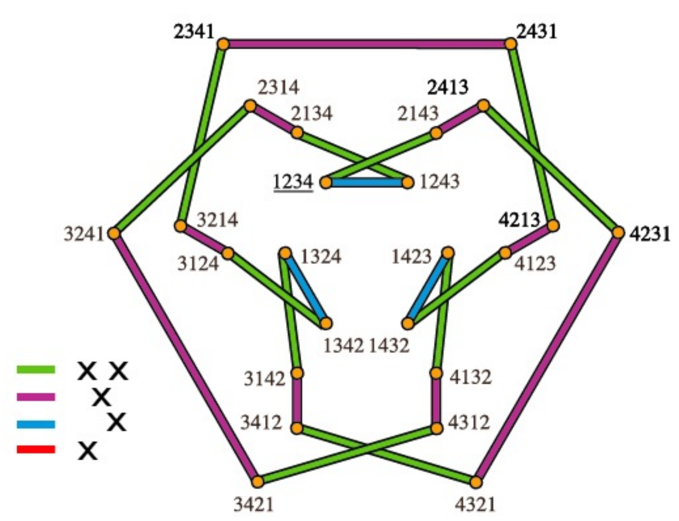


Campanology

Campanology is the art of bell ringing. English mathematicians realised that there was a relationship between bell ringing (Plain Bob Minimus) and Cayley graphs. Since the vertices of a Cayley graph of S_n represent all represent all possible bell ringing permutations of n bells, finding a certain path (called a *Hamiltonian Circuit*) in the graph would result in a change in pattern.

For example:

Plain Bob Minimus is a permutation from bell ringing of 1234 rounds. These are the elements of S_4, the symmetric group of four elements.



Rubik's Cube

The essense of a 2x2x2 Rubik's Cube is a Cayley graph, C_G.

Cayley's Mouse Trap

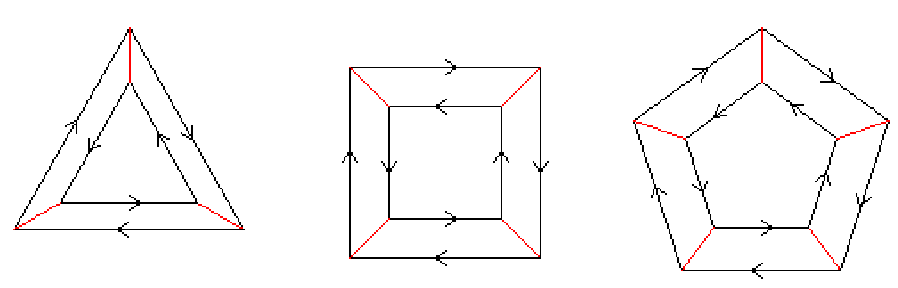
Card game introduced by Cayley based on permutations of 13 cards.

Cayley Digraph

Properties:

- Graph is connected.
- At most one arc goes from a vertex g to vertex h
- Each vertex g has exactly one arc of each type starting at g and one of each type ending at g
- If two different sequences of arc types staring from vertex g lead to the same vertex h then those same sequences of arc types starting from any vertex u will lead to the same vertex v

Examples of Cayley Graphs: D_6, D_8, D_{10}



References

Wikipedia
Britannica.com
web.williams.edu
Dr. Rachel Quinlan, NUIG