The Symmetrical group of a regular, solid tetrahedron.

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MA3343 Group Project

Background

In MA3343 we looked at symmetrical groups of regular 2-Dimensional objects.In this project we are extending the concept of 'symmetry' to a solid 3-Dimensional object: the regular tetrahedron.This platonic solid has four vertices:(A,B,C,D), four faces and six edges:(|AB|,|AC|,|AD|,|BD|,|CB| and |CD|.

Vertices and isomorphism with S4:

There are 4!= 24 permutations of the letters (A,B,C,D), just as there are 24 permutations of the tetrahedron's 4 vertices meaning the group of symmetries we are looking at is isomorphic to S4 which we have studied in class.
There are D4 = 9 possible derangements of these 4 objects, corresponding to the possible permutations involving 'no fixed vertices'.There are 3!=6 possible permutations which leave two objects/vertices in their place and 4c4 = 1 ways of leaving all the vertices as they are ,corresponding with our 'identity element' (A,B,C,D).This leaves us with 4!-3!-D4-1=8 permutations which leaves 1 vertex as it is while rotating the others.The group of 24 symmetries consists of 4P2= 12 distinct reflections and 12 rotations which we are going to explore.

The 'regular' Reflections:

These 6 elements of the group are obtained by reflecting the tetrahedron through each of its 6 edges.These all have exactly 2 'fixed points' each:

- |AB|=(A,B,D,C) -> vertex (A) and (B) are fixed.
- |AC|=(A,D,C,B) -> vertex (A) and (C) are fixed.
- |AD|=(A,C,B,D) -> vertex (A) and (D) are fixed.
- |BD|=(C,B,A,D) -> vertex (B) and (D) are fixed.
- |BC|=(D,B,C,A) -> vertex (B) and (C) are fixed.
- |CD|=(B,A,C,D) -> vertex (C) and (D) are fixed.

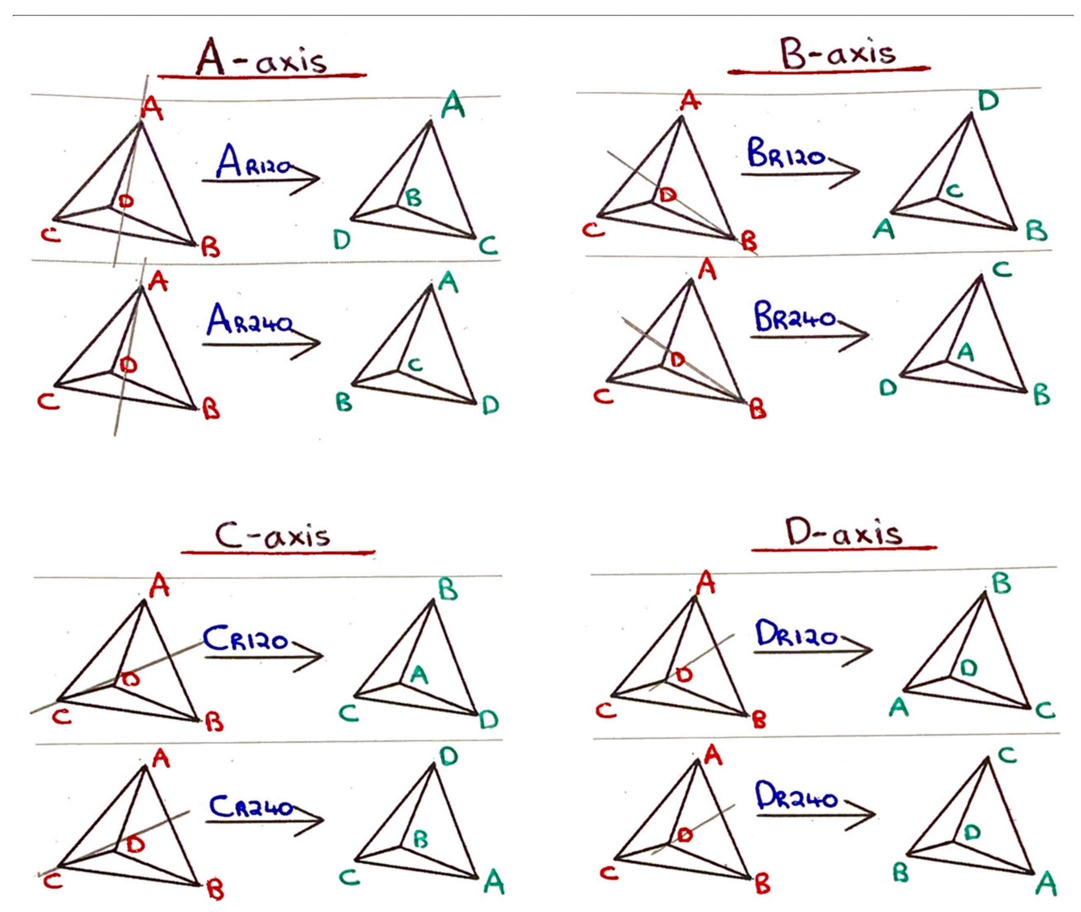
The 'irregular' reflections:

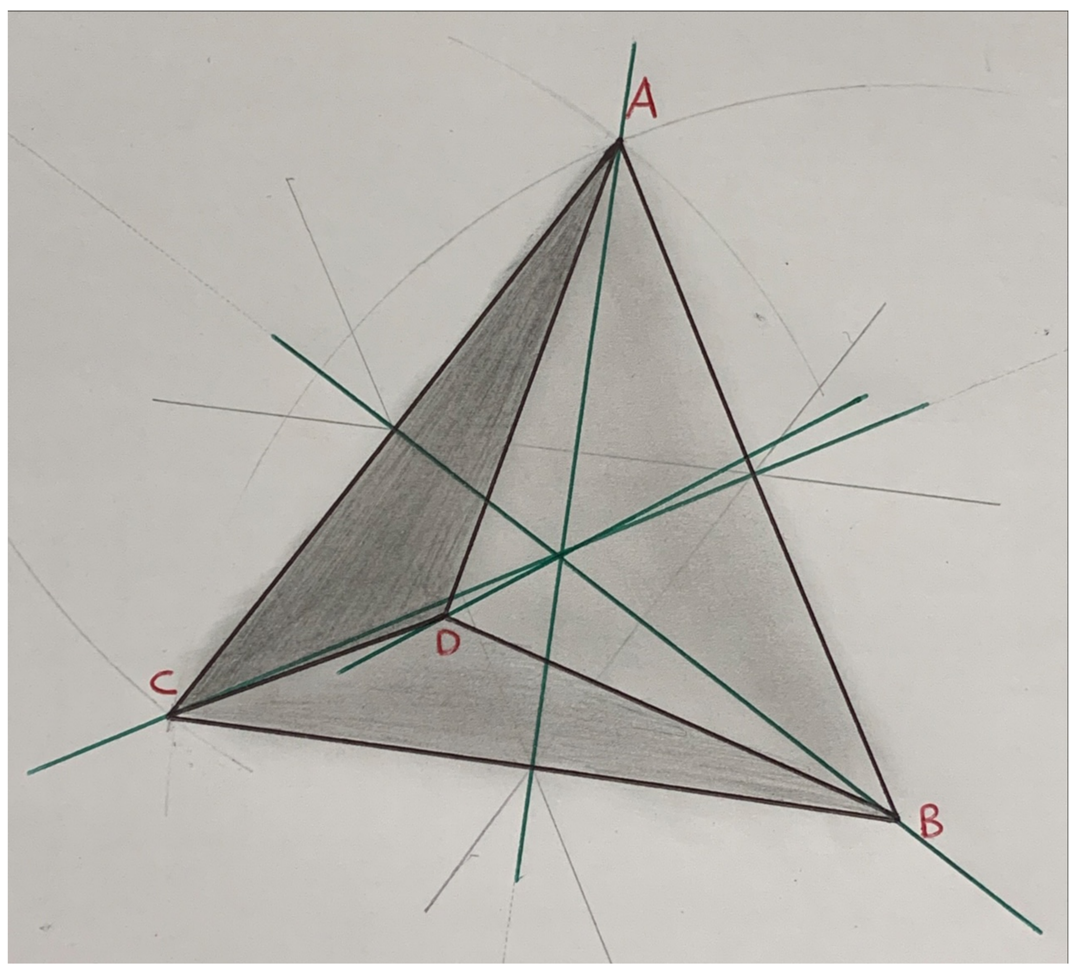
To obtain the remaining 6 permutations, and hence generate the whole 24 elements in the group of permutations of S4, we look at the 6 remaining reflections, which are all derangements.This is done by composing 'regular' reflections and rotations. We did this by 'brute force'and it took a while as you may imagine..

- (B,D,A,C)= Br($\frac{2\pi}{3}$)o|DC|
- (D,A,B,C)= |BC|oAr($\frac{2\pi}{3}$)
- (B,C,D,A)= Ar($\frac{2\pi}{3}$)O|BC|
- (C,A,D,B)= |CD|oBr($\frac{2\pi}{3}$)
- (D,C,A,B)= |AD|oBr($\frac{2\pi}{3}$)
- (C,D,B,A)= Cr($\frac{2\pi}{3}$)o|BD|

So why aren't there images of our beloved reflections? No reflection is possible given we are looking at a platonic SOLID. If we were to do so, it would distort the object and be physically impossible to do. Given the isomorphism with S4 , it is interesting to note that if the tetrahedron was not solid in stature , we would indeed be able to carry out these reflections in the 3-Dimensional plane.

The 'regular' Rotations:

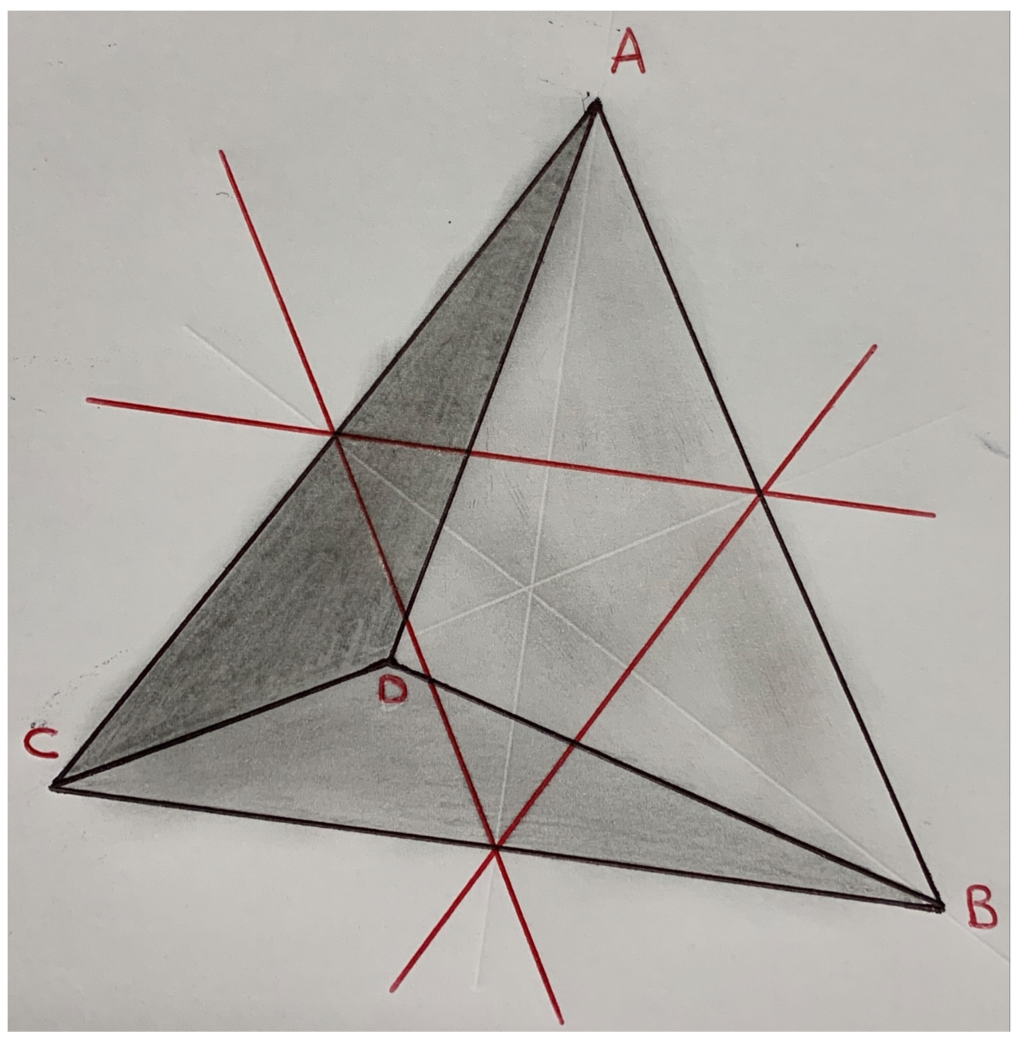


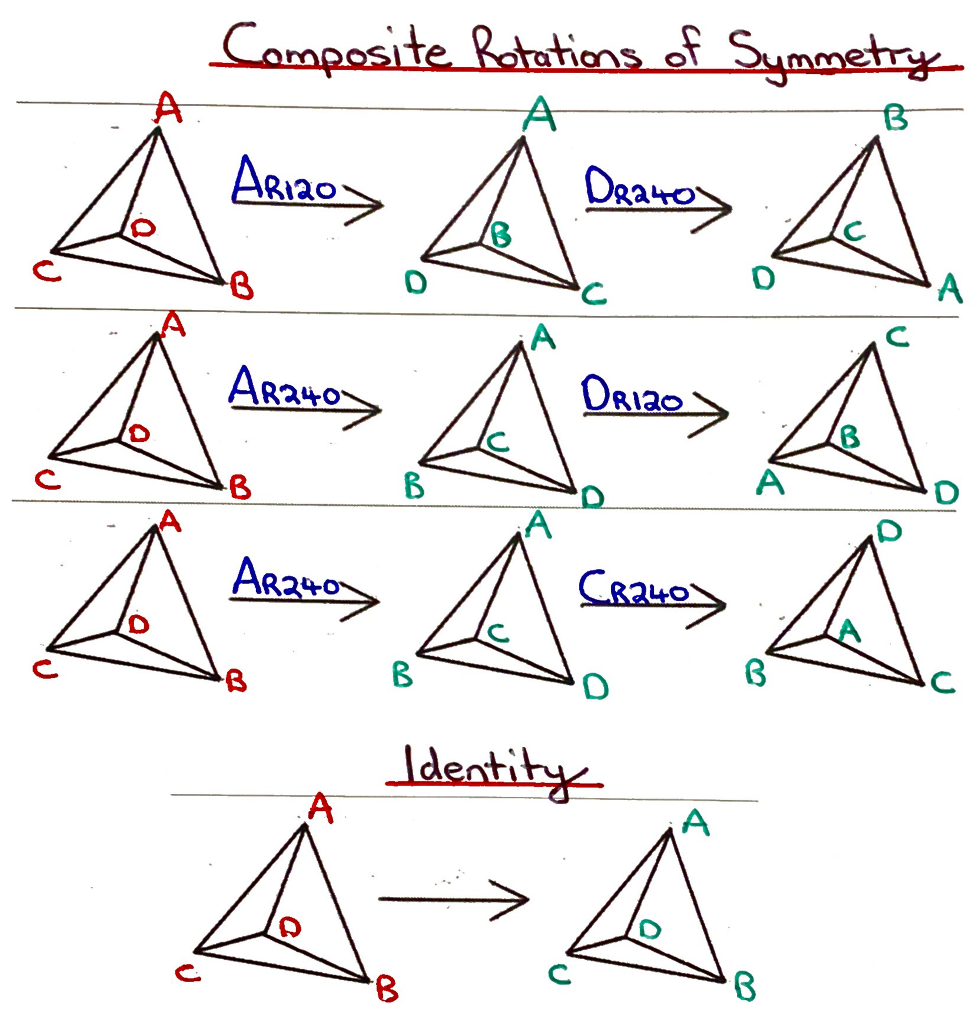


- The 8 'regular'rotations rotate the tetrahedron by and ($\frac{2\pi}{3}$) and ($\frac{4\pi}{3}$) through the axis created by drawing a straight line from the centroid of each face to it's perpendicular vertex, therefore using all four vertices'.These 8 rotations all fix one point (the vertex being used) and correspond to 8 of the 24 permutations of (A,B,C,D), which leave exactly 1 point fixed.

The 'composite' Rotations:

- We obtain the 3 remaining rotations through rotational compositions in different planes.These corrospond to 3 of the 9 possible derangements of the four objects/vertices:





- (B,A,D,C)= Dr($\frac{4\pi}{3}$)oAr($\frac{2\pi}{3}$).–>To visualise these 'irregular' rotations, we pick an edge and
- (C,D,A,B)= Dr($\frac{2\pi}{3}$)oAr($\frac{4\pi}{3}$). place a straight line from the edge's midpoint to the midpoint of
- (D,C,B,A)= Cr($\frac{4\pi}{3}$)oAr($\frac{4\pi}{3}$). the other 3 edeges,creating 3 new axis of rotation,as seen above.
- These 3 'composite' rotations all comprise of a rotation by 180 degrees in each of the 3 axis (red lines in picture above).As mentioned, these account for 3 of the 9 possible derangements of the group S4 (A,B,C,D).All of the 'regular' rotations have 2 fixed points and so, when they are composed , their product will give us a permutation we already have i.e a standard rotation which has 1 fixed vertex or else a rotation with no 'fixed points'.

To Summarise...

- The generating set consists of any two regular rotation that are not on the same axis or the same degree of rotation. e.g (Ar120,Br240)

Extending the concept of group symmetries to a 3D object is a quite interesting.It's perhaps obvious isomorphism with s4 makes the task easier than one would imagine.The link between the tetrahedron's 4 vertices and the elements in the group S4 also makes this very relevant to the group theory we study in class. This mathematical link kept the 'practicality' element to our project.However,the link between conceptual practicality and real world practicality is often a grey area.This is best illustrated by the failure to visually present any reflections. Not for want of trying,but as mentioned earlier it's physically impossible to reflect a platonic solid of any description without compromising the very fact that is is a solid object.So,From a conceptual viewpoint we have 24 elements in the group, but in actual fact we ended up with just 12 (all the possible rotations).