The Formulation of the Group Concept

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A Combination of Major Mathematical Areas

The study of the development of a concept such as that of a group has certain difficulties. For Example could we assume that since the non-zero rationals form a group under multiplication then the origin of the group concept must go back to the beginnings of mathematics. Rather we must take the view that group theory is the abstraction of ideas that were common to a number of major areas which were being studied essentially simultaneously. The three main areas that were to give rise to group theory are:-

- geometry at the beginning of the 19th Century,
- number theory at the end of the 18th Century,
- the theory of algebraic equations at the end of the 18th Century leading to the study of permutations.

Geometry: Monge, Carnot and Poncelet

Geometry has been studied for a very long time so it is reasonable to ask what happened to geometry at the beginning of the 19th Century that was to contribute to the rise of the group concept:

Lazare Carnot 1746 - 1818

Gaspard Monge 1746 - 1818

Jean Poncelet 1788 - 1867

Losing its 'metric' character: Gaspard Monge

The four memoirs that Monge submitted to the Académie were on a generalisation of the calculus of variations, infinitesimal geometry, the theory of partial differential equations, and combinatorics. Over the next few years he submitted a series of important papers to the Académie on partial differential equations which he studied from a geometrical point of view. His interest in subjects other than mathematics began to grow and he became interested in problems in both physics and chemistry.
Although Monge eventually moved away from Mathematics he left a great contribution to the formulation of the group concept through his study of projective and non-euclidean geometries

The Abstraction of Geometry: Lazare Carnot and Jean Poncelet

Despite his Engineering background. Lazare Carnot is best known as a geometer. In 1801 he published *De la correlation des figures de géométrie.* in which he tried to put pure geometry into a universal setting. He showed that several of the theorems of Euclid's Elements can be established from a single theorem.
In 1803 he published *Géométrie de position* in which sensed magnitudes were first used systematically in geometry. This work greatly extended his work of 1801 and in it Carnot again shows what quantities mean to him writing:- *"Every quantity is a real object such that the mind can grasp it or at least its representation in calculation".*

Perhaps the most work done in showing the difference between metric and incidence geometry comes from the work of Poncelet, greatly influenced by the work of Monge and Carnot He published Traité des propriétés projectives des figures in 1822, which is a study of those properties which remain invariant under projection. This work contains fundamental ideas of projective geometry such as the cross-ratio, perspective, involution and the circular points at infinity.Poncelets most recognised for is his theorem which states that whenever a polygon is inscribed in one conic section and circumscribes another one, the polygon must be part of an infinite family of polygons that are all inscribed in and circumscribe the same two conics. This theorem was then interpreted using group theory and the complex plane.

Möbius' Unintentional Contribution

Möbius in 1827, although he was completely unaware of the group concept, began to classify geometries using the fact that a particular geometry studies properties invariant under a particular group. Steiner in 1832 studied notions of synthetic geometry which were to eventually become part of the study of transformation groups.

Möbius
August Möbius is best known for his work in topology, especially for his conception of the Möbius strip, a two dimensional surface with only one side.
Not only a great mathematician Möbius did publish important work on astronomy concerning occultations of the planets He also wrote on the principles of astronomy.

Number Theory: Euler and Gauss

Euler and the Abelian Group

In 1761 Euler studied modular arithmetic. In particular he examined the remainders of powers of a number modulo n. Although Euler's work is, of course, not stated in group theoretic terms he does provide an example of the decomposition of an abelian group into cosets of a subgroup. He also proves a special case of the order of a subgroup being a divisor of the order of the group.

Proving Eulers theorem using Group Theory

Eulers theorem

$a^{\varphi(n)} \equiv 1 \pmod{n}$

where $a^{\varphi(n)} isEulerstotentfunction$

The residue classes modulo n that are coprime to n form a group under multiplication. The order of that group is (n) where denotes Euler's totient
Lagrange's theorem states that the order of any subgroup of a finite group divides the order of the entire group, in this case (n).
If a is any number coprime to n then a is in one of these residue classes, and its powers a, a2, ... , ak are a subgroup modulo n, with ak 1 (mod n). Lagrange's theorem says k must divide (n), i.e. there is an integer M such that kM = (n). This then implies
$a^{\varphi(n)} = a^{kM} = (a^k)^M \equiv 1^M = 1 \equiv 1 \pmod{n}$.

Gauss and the continuation of Eulers work

Gauss in 1801 was to take Euler's work much further and gives a considerable amount of work on modular arithmetic which amounts to a fair amount of theory of abelian groups. He looked at binary quadratic forms:

ax2 + 2bxy + cy2 where a, b, c are integers.

Gauss examined the behaviour of forms under transformations and substitutions. He partitions forms into classes and then defines a composition on the classes. Gauss proves that the order of composition of three forms is immaterial so, in modern language, the associative law holds. In fact Gauss has a finite abelian group and later (in 1869) Schering, who edited Gauss's works, found a basis for this abelian group.

Gauss
Gauss has been described as "the greatest mathematician since antiquity", having had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians

Contributers to the study of permutations

Lagrange

Permutations were first studied by Lagrange in his 1770 paper on the theory of algebraic equations. Lagrange's main object was to find out why cubic and quartic equations could be solved algebraically.
In studying the cubic, for example, Lagrange assumes the roots of a given cubic equation are x', x" and x"'. Then, taking 1, w, w2 as the cube roots of unity, he examines the expression

R = x' + wx" + w2x"'

and notes that it takes just two different values under the six permutations of the roots x', x", x"'.
Although the beginnings of permutation group theory can be seen in this work, Lagrange never composes his permutations so in some sense never discusses groups at all.

Ruffini

The first person to claim that equations of degree 5 could not be solved algebraically was Ruffini. In 1799 he published a work whose purpose was to demonstrate the insolubility of the general quintic equation. Ruffini's work is based on that of Lagrange but Ruffini introduces groups of permutations
Ruffini divides his *permutazione* into types, namely *permutazione semplice* which are cyclic groups in modern notation, and *permutazione composta* which are non-cyclic groups

Cauchy

Cauchy played a major role in developing the theory of permutations publishing a major work which sets up the theory of permutations as a subject in its own right
He introduces the notation of powers, positive and negative, of permutations (with the power 0 giving the identity permutation), defines the order of a permutation, introduces cycle notation and used the term *système des substitutions conjuguées* for a group

Modern day Group theory

Group theory really came of age with the book by Burnside Theory of groups of finite order published in 1897. The two volume algebra book by Heinrich Weber (a student of Dedekind) Lehrbuch der Algebra published in 1895 and 1896 became a standard text. These books influenced the next generation of mathematicians to bring group theory into perhaps the most major theory of 20th Century mathematics.