The Group of Symmetries of the Cube (Rotations)

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Outline

The aspects of the cubes symmetries we will investigate are:

- Verifying the rotations of a cube form a group.
- Displaying how this group complies with the orbit stabilizer theorem.
- Showing the symmetry group of a cube is isomorphic to S_4.
- Finding subgroups and proving for Lagrange's theorem.

Rotations of a cube form a group.

- Closure.

Let, G be the group of symmetries in a cube and let $a, b \in G$. Need to show $a * b \in G$. We can prove this by showing $x * (a * b) = (x * a) * b$. Clearly any 3 rotations applied in the same order, on two different occasions, give the same result. Therefore the rotations of a cube obeys associativity. If $a \in G$ then applying a to an unmarked cube gives a seemingly identical looking cube. If $b \in G$ and applied to this result, we still appear to have an identical looking cube. Thus $a * b \in G$.

- Identity.

Does an id element exist within the rotations of cube such that: $id * x = x * id, x = x, \forall x \in G$ The identity is any rotation that leaves the cube completely unchanged. This is R_{360} through any axis.

- Inverse.

Each rotation is its own inverse. For example, take a rotation $x \in G$. If you apply the rotation x and then apply the same rotation in the the opposite direction, you end up where you originally started. Thus $x^{-1} \in G$.

Orbit-Stabilizier Theorem

f = face in a cube
$|G \cdot f| = |G : Stab_G(f)|$
(Note: There are 24 rotational symmetries in a cube i.e. $|G| = 24$).

Proof.

Take any face on a cube. It is possible to move from that face to any face in the cube (see figure 1, right). So the orbit of any face is $\{1, 2, 3, 4, 5, 6\}$ (where each number is a face in the cube). The stabilizer of all faces in cube are $\{id, R_{90}, R_{180}, R_{270}\}$ (see figure 1 and apply appropriate axis).
$|G \cdot f| = 6, |G : Stab_G(f)| = |24 : 4| = 6$
$6 = 6$ □

e= edge in a cube
$|G \cdot e| = |G : Stab_G(e)|$

Proof.

Take any edge on a cube and again it's possible to move from that edge to any edge in the cube. So the orbit of any edge is $\{1, 2, 3, ..., 12\}$ (each number is an edge in the cube. The stabilizer of any edge is $\{id, R_{180}\}$ (where the axis of rotation goes from the centre of that edge to the centre of the edge on the face directly opposite, passing through the centre of the cube. See figure 3, right).
$|G \cdot e| = 12, |G : Stab_G(e)| = |24 : 2| = 12$
$12 = 12$ □

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Rotations[1]

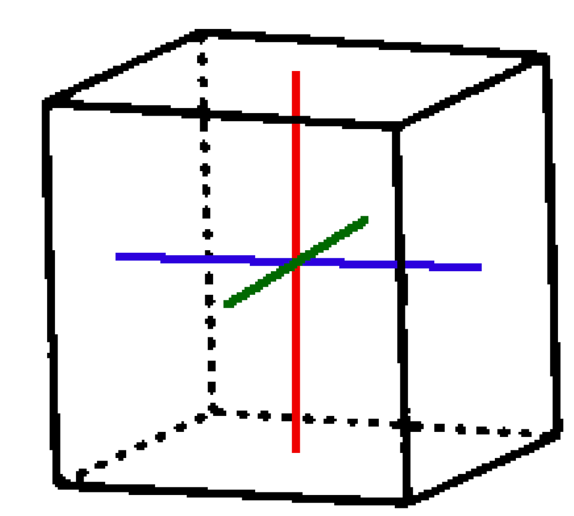

Figure 1:Face Midpoint Rotation.

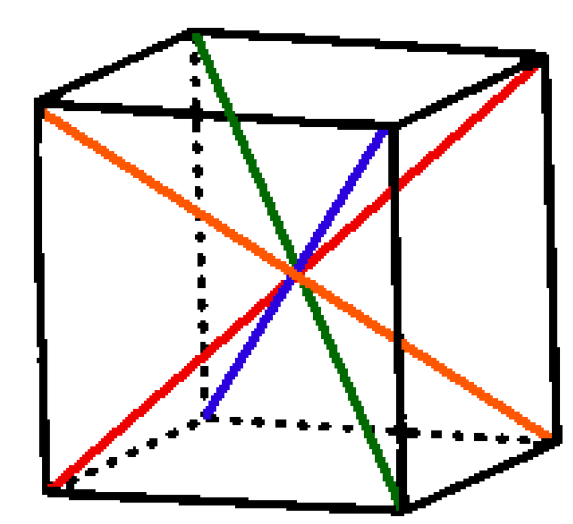

Figure 2:Diagonal Rotation.

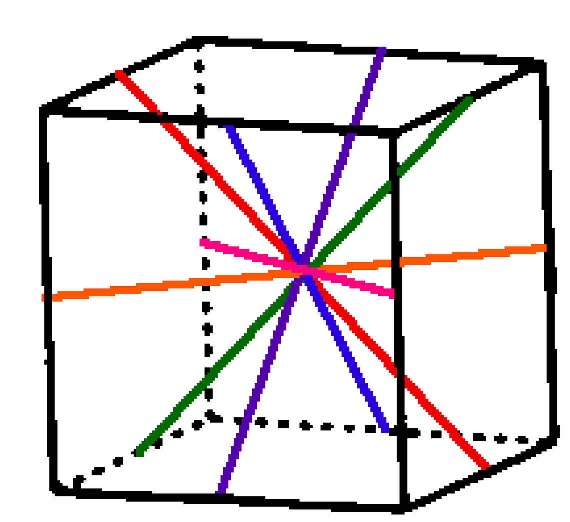

Figure 3:Edge Midpoint Rotation.

Abelian

$Z(G) = \{id\}$
$C_G(x) = \{id$, any rotation on the same axis as $x : \forall x \in G\}$
Hence the group is not abelian.
The subgroups mentioned on the right are abelian as rotations on the same axis commute with each other.

Isomorphic to S_4

- Identity = 1.
  Permutations in S_4: $(1)(2)(3)(4)$, R_{360} through any axis.
- Edge Midpoint Rotation = 6
  Permutation in S_4:$(12)(13)(14)(23)(24)(34)$, (see figure 3).
- Diagonal Rotation = 8
  Permutation in S_4:
  $(123), (124), (134), (132), (142)(143), (234), (243)$, (see figure 2).
- Face Midpoint Rotation (horizontal axis) = 6
  Permutation in S_4:
  $(1234), (1243), (1324), (1342), (1423), (1432)$, (see figure 1).
- Face Midpoint Rotation (vertical axis) = 3
  Permutation in S_4: $(12)(34), (13)(24), (14)(23)$, (see figure 1).

Subgroups

- Stabilizer of each face forms a subgroup of order 4 and stabilizer of each edge forms a subgroup of order 2.
- For axis shown in figure 1, all rotations through the same axis form a subgroup of order 4.
- For axis shown in figure 2, all rotations through the same axis form a subgroup of order 3.
- For axis shown in figure 3, all rotations through the same axis form a subgroup of order 2.

Order of the subgroups are 2,3 and 4 and are factors of $|G| = 24$.Verifying Lagrange's theorem.

References

[1] *Rotation Diagrams*.
http://garsia.math.yorku.ca/zabrocki/math4160w03/cubesyms/.