Rotational Symmetries Of The Cube

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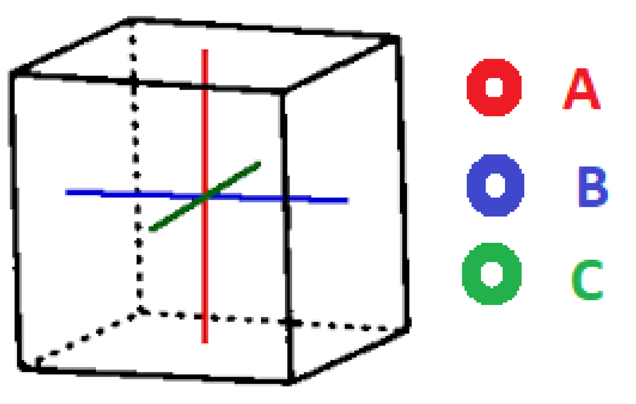


Figure 1: *Cube 1 has 9 rotations*

1. The Cubes Rotation On Faces

There are 9 possible rotations of the cube about the three axes shown in Figure 1 above:

- 1. Axis A
- 2. Axis B
- 3. Axis C

  We can either rotate by 90°, 180° or 270° around either the red, blue or green axes. Each of these rotations will leave two faces fixed, as the axes will go through the centres of two opposite faces. All vertices and edges are not fixed. When you work out the cycle structure of these 9 rotations, notice that the 6 rotations by 90° or 270° will have the same cycle structure and the other 3 rotations of 180°about each of the axes will have the same cycle structure.

Example

If we choose the line A as an example axis as shown, we can clearly see the axis can rotate the through 90°, 180° an 270°. If we rotate through 0° on Axis A, all faces, vertices and edges will be fixed.
If we rotate anti-clockwise through 90° on Axis A:
(W) moves to (Z)
(X) moves to (W)
(Y) moves to (X)
(Z) moves to (Y)

If we rotate anti-clockwise through 180° on Axis A:
(W) moves to (Y)
(X) moves to (Z)
(Y) moves to (W)
(Z) moves to (X)

If we rotate anti-clockwise through 270° on Axis A:
(W) moves to (X)
(X) moves to (Y)
(Y) moves to (Z)
(Z) moves to (W)

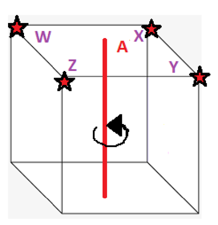


Figure 2: *Example of cube with Axis A going through two opposite faces.*

2. Rotation by 180°

Rotations about an axis from the center of an edge to the center of the opposite edge

- There are six axes from the center of an edge to the center of the opposite edges shown in figure 2-U,V,W,X,Y,Z
- We can only rotate these axis around 180°
- each axis has one rotation each
- Each of these rotations leave two edges fixed and no faces or vertices fixed.
- Each of these 6 permutations will have the same cycle structure.

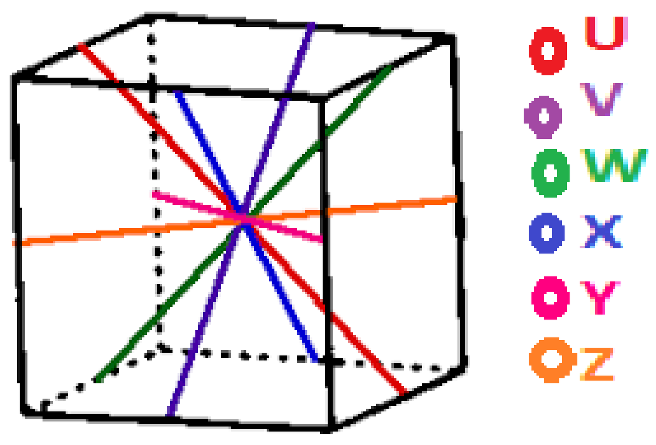
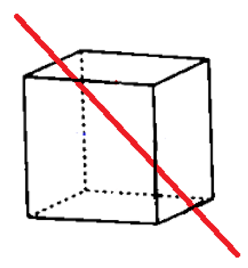


Figure 3: *Cube 2 has 6 rotations*

Example

- if we choose the line U as an example axis as shown
- We can clearly see the axis can only rotate once, through 180°



3. The identity Rotation

- Identity symmetry is a basic rigid motion that maps a figure back onto itself.
- The cube also has the identity as a rotational symmetry.

4. The Cubes Rotation On Corners

The Four Axes And Their Rotations

There are 8 possible rotations for the axes on the corners. Looking at Figure 5 you can see 4 axes M, N, L and O in each of the four corners. If you pick a corner as a fix point, there are three edges going away from that point. This means, if you pick another point at the end of one of those edges and rotate it, it has only two possible rotations: You can rotate only by 120° and 240°.
So there are two rotations for each of the four axis. 2 * 4 = 8. Hence, in total there are 8 possible rotations.

5. Examples For Rotations

Examples:

Look at Figure 4 and take A as the starting point and L as the axis that you rotate around.

- When you rotate from A by 120° clockwise around L, you end up at point B.
- When you rotate from A by 240° clockwise around L, you end up at point C.

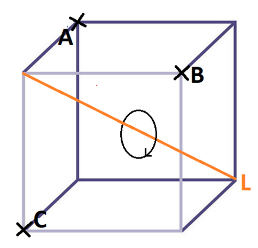


Figure 5: *Logo of the cube with axes in the corner.*

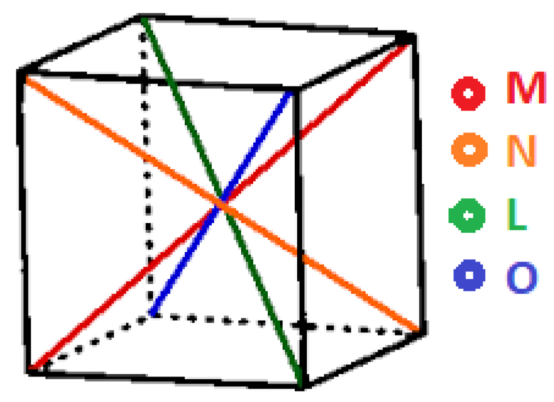


Figure 6: *Cube 3 has 8 rotations.*

6. Conclusion

We can see from cube 1, 2 and 3 together with the identity element that the group of rotations for the cube has a total of 24 elements, 23 rotations and 1 id.

7. Compare to S4

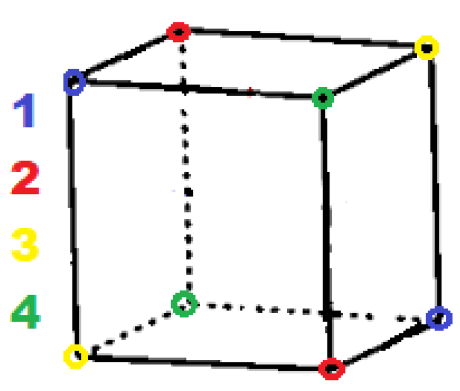


Figure 7: *Example of cube with Axis A going through two opposite faces.*

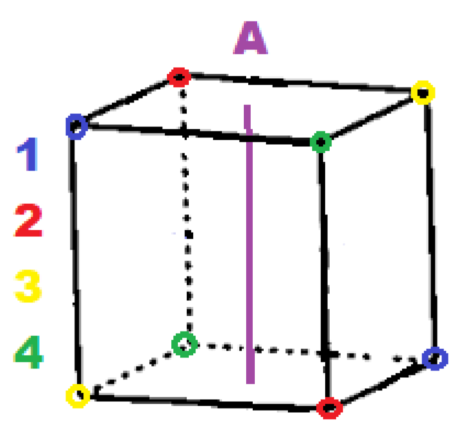


Figure 8: *Example of cube with Axis A going through two opposite faces.*

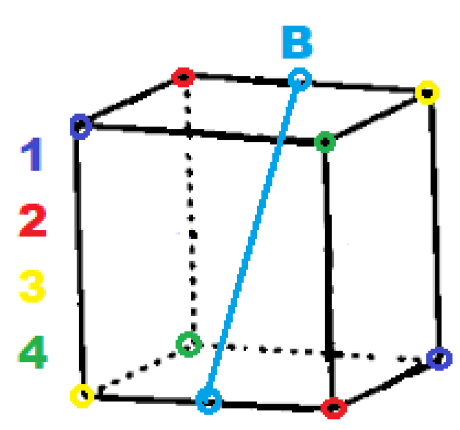


Figure 9: *Example of cube with Axis B going through the centre of two opposite edges.*

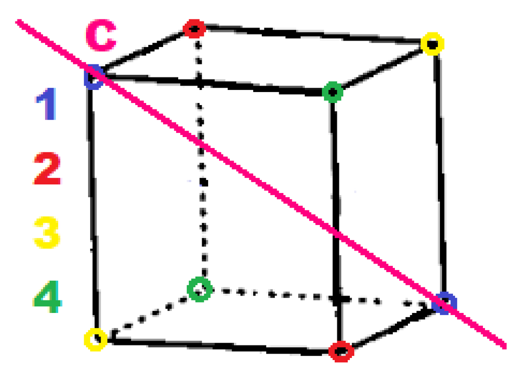


Figure 10: *Example of cube with Axis C going through two opposite corners.*

S4 is the group of permutation for 4 objects. S4 = 1,2,3,4. Those objects refer to the pairs of the opposite corners in the cube (see figure 7).

- Example 1: Axis with faces. In this example we are doing the axis of rotation through axis A (see figure 8).
  If we do a clockwise rotation of 90° through the A axis, we get: R90°*1 = 2, R90°*2 = 3, R90°*3 = 4, R90°*4 = 1. Hence R90° correspondences with (1 2 3 4).
  If we do a clockwise rotation of 180° through the A axis, we get: R180°*1 = 3, R180°*2 = 4, R180°*3 = 1, R180°*4 = 2. Hence R180° correspondences with (1 3)(2 4).
  If we do a clockwise rotation of 270° through the A axis, we get: R270°*1 = 4, R270°*2 = 1, R270°*3 = 2, R270°*4 = 3. Hence R270° correspondences with (1 4 3 2).
- Example 2: Axis with edges. In this example we are doing the axis of rotation through axis B (see figure 9).
  If we do a clockwise rotation of 180° through the B axis, we get: R180°*1 = 1, R180°*2 = 3, R180°*3 = 2, R180°*4 = 4. Hence R180° correspondences with (1)(2 3)(4).
- Example 3: Axis with corners. In this example we are doing the axis of rotation through axis C (see figure 10).
  If we do a clockwise rotation of 120° through the C axis, we get: R120°*1 = 1, R120°*2 = 4, R120°*3 = 2, R120°*4 = 3. Hence R120° correspondences with (1)(2 4 3).
  If we do a clockwise rotation of 240° through the C axis, we get: R240°*1 = 1, R240°*2 = 3, R240°*3 = 4, R240°*4 = 2. Hence R240° correspondences with (1)(2 3 4).

This shows that the cube's rotational symmetries permute all the objects in S4 in all possible ways.

Presented at the "Title of the Conference", City, Country, Year