The History of Lagrange's Theorem

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Introduction

In group theory, there is a well-known theorem defining the correlation between the order of a group and the order of said groups subgroup. This theorem is called Lagrange's theorem and is named after the Italian mathematician Joseph Louis Lagrange. However, Lagranges original theorem has changed over time and actually pre-dates the concept of a group. The poster is an attempt to connect and explore the original theorem of Lagrange, and the theorem we know today.

Modern Day Lagrange's Theorem

In the case of a finite group, Lagrange's theorem gives us insight about what sub-groups are and aren't possible.

- If G is a group and H is a sub-group of G, then the order of every sub-group H of G divides the order of G.

Lagrange's theorem allows someone to recognise what set could be a subgroup
i.e. A group of order 12 could not have a sub-group order 7

The Man, The Myth, The Legend

Born in Turin on the 25th January1736, Giuseppe Luigi Lagrangia made contributions to many areas of mathematics including analysis, number theory and mechanics. Several area's of mathematics bear his name as testament to his legacy such as Lagrangian in mechanics and the Euler-Lagrange equation in calculus. He died 10 April 1813 in Paris

Proof of Lagrange's Theorem

The Proof of Lagrange's Theorem relies on 3 Lemmas which are as follows

- 1. If G is a group with subgroup H, then there is a one to one correspondence between H and any coset of H
- 2. If G is a group with subgroup H, then the left coset relation, $g1 \sim g2$ if and only if $g1 * H = g2 * H$ is an equivalence relation.
- 3. Let S be a set and $\sim$ be an equivalence relation on S. If A and B are two equivalence classes with $A \cap B \neq \theta$, then $A = B$.

Let $\sim$ be the left coset equivalence relation defined in Lemma 2. It follows from Lemma 2 that $\sim$ is an equivalence relation and by Lemma 3 any two distinct cosets of $\sim$ are disjoint.
Hence, we can write

$$G = (g1 * H) \cup (g2 * H) \cup \bullet\bullet\bullet \cup (g\ell * H)$$

where the $gi * H, i = 1, 2, \ldots \ell$ are the disjoint left cosets of H guaranteed by Lemma 3. By Lemma 1, the cardinality of each of these cosets is the same as the order of H, and so

$$|G| = |g1 * H| + |g2 * H| + \bullet\bullet\bullet + |g * H|$$
$$= |H| + |H| + \bullet\bullet\bullet + |H|$$
$$\ell \text{ summands} = \ell * |H| = \ell * k.$$

The Theorem of Lagrange

- If a function $f(x1, x2...xn)$ at $n!$ variables is acted on by all possible permutations of the variables and these permuted functions take only r distinct values then r is a division of n!.

Lagrange came across this result while trying to find a formula solution for a Quintic polynomial and more generally a formula for nth degree polynomial where $n > 4$. Lagrange observed that the solution for the cubic and quaratic equations could be solved by finding an equation of a lower degree, this equation is called a "resolvent" for example, the quartic was solved using a cubic resolvent polynomial whose roots could be written as

$$\frac{x_1x_2 + x_3x_4}{2}, \frac{x_1x_3 + x_2x_4}{2}, \frac{x_1x_4 + x_2x_3}{2}$$

where $x1, x2, x3, x4$ are roots of the correspondent polynomial. When these 4 roots are permeated in everyone of the 24 permeatations, only these 3 usually occurs. In order to use the method to solve a quartic, Lagrange proposed we would need to find a function which only takes 4 values when the variable is permuted in 120 (5!) ways.

From Polynomials to Group Theory

The work of Lagrange would remain relevant over the years following its publication in 1770. Paolo Ruffini would take Lagrange's work and further it. This includes showing that the converse of what we know today as Lagrange's theorem is false. A proof of the "Theorem of Lagrange" would come from Abbati in 1802. This marks the first time a complete proof had been given.

In 1815, 2 years after Lagrange's death, Augustin-Louis Cauchy would publish a paper which furthered permutation group theory as an independent topic of mathematics. However, the notion of a group did not appear in it.

Évariste Galois introduced the term group for permutation groups in a paper on solutions of polynomials by radicals in 1831. In 1844, Cauchy furthered the notion of group theory, Giving us the main axioms that are used today. He also proved Lagrange's theorem for symmetric groups.

Cauchy published several shorter papers over the following years, showing the connection between the "Theorem of Lagrange" and "Lagrange's Theorem". It was shown that "Theorem of Lagrange" is essentially "Lagrange's Theorem" for the case where $G = Sn$

Although a number of authors credited the theorem to Lagrange, many didn't mention Lagrange's name and it was some time before it became widely known as "Lagrange's Theorem." Van der Waerden's "Moderne Algebra" was one of the most influential texts in algebra. It first appeared in 1930. Some 117 years after his death.

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References - Richard L. Roth. "A History of Lagrange's Theorem on Groups". In:Mathematics Magazine74.2 (2001), pp. 99–108.