Rubik's Cube's Applications to Group Theory

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Axioms of a Group

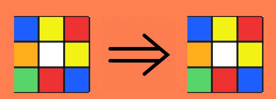
Rubik's cubes satisfy all the conditions of a group:

- Contains an Identity Element
- Inverse Element
- Closure
- Associativity

1. Identity Element

An identity element ξ for a binary operation $*$ is one that has no effect on any element when combined with that element using $*$, it is "neutral".

Labelling M as a random combination of moves and e as the "empty" identity move, M * e = M. Thus e (not moving) is the identity element.

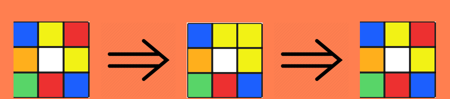


2. Inverse Element

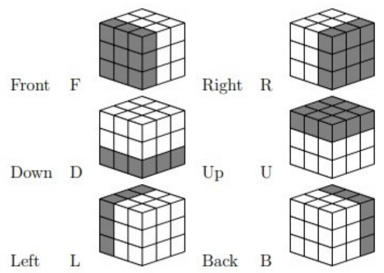
Let X,Y $\in G$. An element X is an inverse element of Y if X$*$Y = Y$*$X = ξ. Where ξ is the neutral element.

Let M be a move and the inverse be M'. Doing these together returns you to your starting point. This is like the "empty move" e - the neutral element.

$\therefore M$*M' = e (ξ)

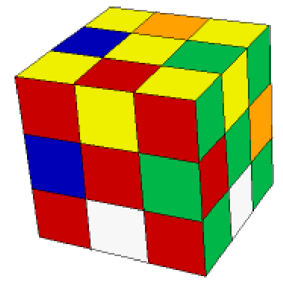


Notation



- We denote the six faces into Front (F), Back(B), Left(L), Right(R), Up(U), Down(D).
- **F** = 90 degree rotation clockwise of the front face.
- **FF** = 180 degree rotation clockwise of the front face.
- **FU** = 90 degree rotation clockwise of the front face followed by 90 degree rotation clockwise of the up face.
- **F'** = 90 degree rotation counter-clockwise of the front face.
- **FF'** = 90 degree rotation clockwise of the front face followed by a 90 degree rotation counter-clockwise of the front face, which results in what you originally had.
- $\therefore$ **FF'** = ξ
- $\therefore$ F' is the inverse of F.

The Centre of the Group?



- For a group G, the center Z(G) is defined as:

$$Z(G) = \{x \in G : xg = gx \; \forall \; g \in G\}.$$

- The centre is the group of elements that commute with the whole group. The combination of Rubik's Cube do not commute with each other - apart from the identity (obviously) and the "Superflip" (s). [Pictured Above]
- The superflip or is a Rubik's Cube configuration in which all 20 of the movable subcubes ("cubies") are in the correct permutation, and the eight corners are correctly oriented, but all twelve of the edges are "flipped". The superflip, unlike all other Rubik's Cube moves, commutes with every possible move, making the centre equal to {e, s}.

Subgroups?

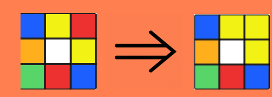
- The number of possible permutations of the squares on a Rubik's cube is given by $\frac{8! * (3^8) * 12 * 2^{12}}{3*2*2}$ which is roughly equal to $4.3252 * 10^{19}$
- Knowing Lagrange's Theorem states that if we let G be a finite group and let H be a subgroup of G. Then the order of H divides the order of G.
- Therefore any subgroup of the Rubik's Group has order of a divisor of $\frac{8! * (3^8) * 12 * 2^{12}}{3*2*2}$
- Eg - the group generated by U and RR has size $(2^6 * 3^2 * 5^2)$. Which divides evenly into the order of G, making it a subgroup of G.

3. Closure

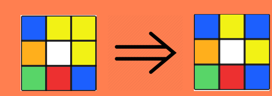
Let M_1 and M_2 be any two moves of a Rubik's Cube, $\therefore M_1$, $M_2 \in G$. In order for G to be closed under the operation $*$, $M_1 * M_2 \in G$.

This is true as no matter which two moves we combine, it will return an acceptable move, meaning it's part of the group. i.e. $M_1 * M_2 \in G$.

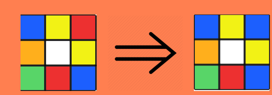
For example, if M_1



If M_2



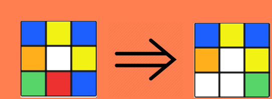
Then $M_1 * M_2$ is:



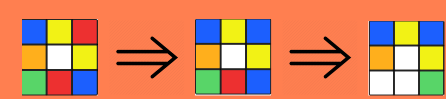
This move would clearly be in the group.

4. Associativity

Let M_1, M_2 and M_3 be any three moves, $\therefore M_1$, M_2 $M_3 \in G$.
For G to be associative:
$(M_1 * M_2) * M_3$ = $M_1 * (M_2 * M_3)$.
This is clearly true; if we do M_1 and M_2 first, then M_3, you'll end up with the same as if you did M_1 first then M_2 and M_3.
Using M_1 and M_2 before and M_3 to be:



Doing $(M_1 * M_2) * M_3$ will result in:



Then $M_1 * (M_2 * M_3)$ will result in: