Generating sets of symmetric groups

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Introduction

A symmetric group is the group of permutations on a set of n elements. It's denoted using the term S_n and is a finite group which contains n! elements. Symmetric groups play an important role in mathematics and can be seen in areas such as combinatorics, invariant theory and Galois theory. In this poster we will be discussing the generating sets of these groups and analysing their properties.

Properties of symmetric groups

Symmetric groups are sets of permutations of elements.

For example:
S_3 =
(1 2 3),
(2 1 3),
(3 2 1),
(1 3),
(2 3 1),
(3 1 2)

as we can see the set has 6 = 3! elements In general Symmetric groups always have n! elements . This is because there are n! permutations of n elements .

Generating sets of size n for the symmetric Groups

- A transposition is a permutation that interchanges two elements and leaves all the others fixed.
- The set of all transpositions of adjacent elements, viz permutations which exchange i with i+1 and leave the remaining elements unchanged, is a generating set for the symmetric group.
- Further it is true that given any element of the symmetric group:
- it can be expressed as a product of length at most $\binom{n}{k}$ of these generators.
- this product can be determined algorithmically, using O(n^2)*steps*.

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How Many Permutations are There?

For a set of size n, there are exactly n! permutations.

Proof:

First, observe that the number of permutations of a set S is finite and is dependent only on the cardinality of S.

Further, observe that it is equal to the number of bijections between any two sets of the same cardinality. Call this number f(n).

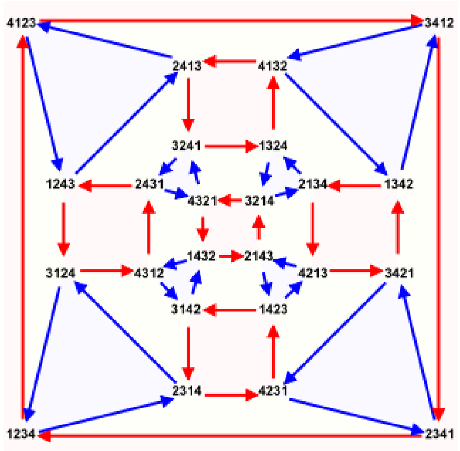
Now pick a $\in$ S. There are n possibilities for $\delta(a)$. Whatever value we choose for $\delta(a)$, there are n-1 possible values for the images of the remaining elements under δ.

Thus, for each choice of $\delta(a)$, there are f(n-1) possibilities.// Thus,

$$f(n) = nf(n-1)$$

And we have f(n) = n!

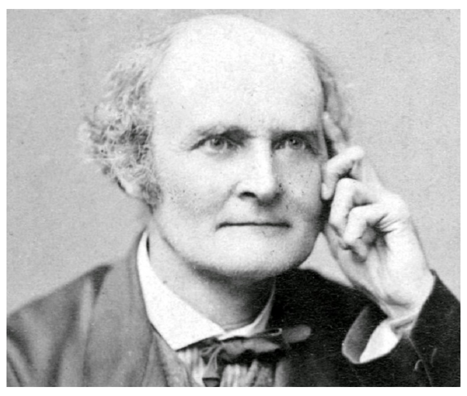
Application : Cayley Graphs



A Cayley Graph of symmetry group S_4, the group of all permutations of 4 elements, with 4! = 24 elements.

Cayley's Theory

This theorem , named after Arthur Cayley, states that every group G is isomorphic to a subgroup of the symmetric group acting on G.



He considers all groups to be permutation groups of some set and therefore deduces that theorems which are true for subgroups of permutation sets are true for all groups.

Theorem

S_n has generating sets with just two elements.

Proof:

This is clear for n=1 and n=2. for n $\geq$ 3,

we note (1) = $(12)^2$ and every cycle of length $>$ 2 is a product of transpositions.

For example,

$$(13526) = (13)(35)(52)(26)$$

Since the cycles generate S_n, and products of transpositions gives us all cycles, the transpositions generate S_n.

This theorem tells us that S_n is generated by elements of order 2.

References

1)https://groupprops.subwiki.org/wiki/Generating$_{s}ets_{f}or_{s}ubgroups_{o}f_{s}ymmetric_{g}roup : S3$
2)https://en.wikipedia.org/wiki/Cayley$_{g}raph$
3)https://www.cmi.ac.in/ vipul/studenttalks/thesymmetricgrouppart2.pdf
4)https://kconrad.math.uconn.edu/blurbs/grouptheory/genset.pdf