Arthur Cayley and his investigation of groups

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Introduction

In 1854 British mathematician Arthur Cayley published a paper titled "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$". In this paper Cayley formulated the first definition of the algebraic structure known as groups.

Cayley's observation

While investigating the solutions to the equation $\theta^n = 1$ Arthur Cayley discovered some interesting properties belonging to the solution set. He noticed that if you multiplied two elements of the solution set you would obtain another element. From this he found the set was closed under the operation of multiplication. Another observation made by Cayley was the fact that if you multiplied two certain elements you would get 1 or 'the identity element', this implied that the multiplicative inverse of each element was part of set. As the solutions were complex numbers the multiplication of three elements was already commutative.

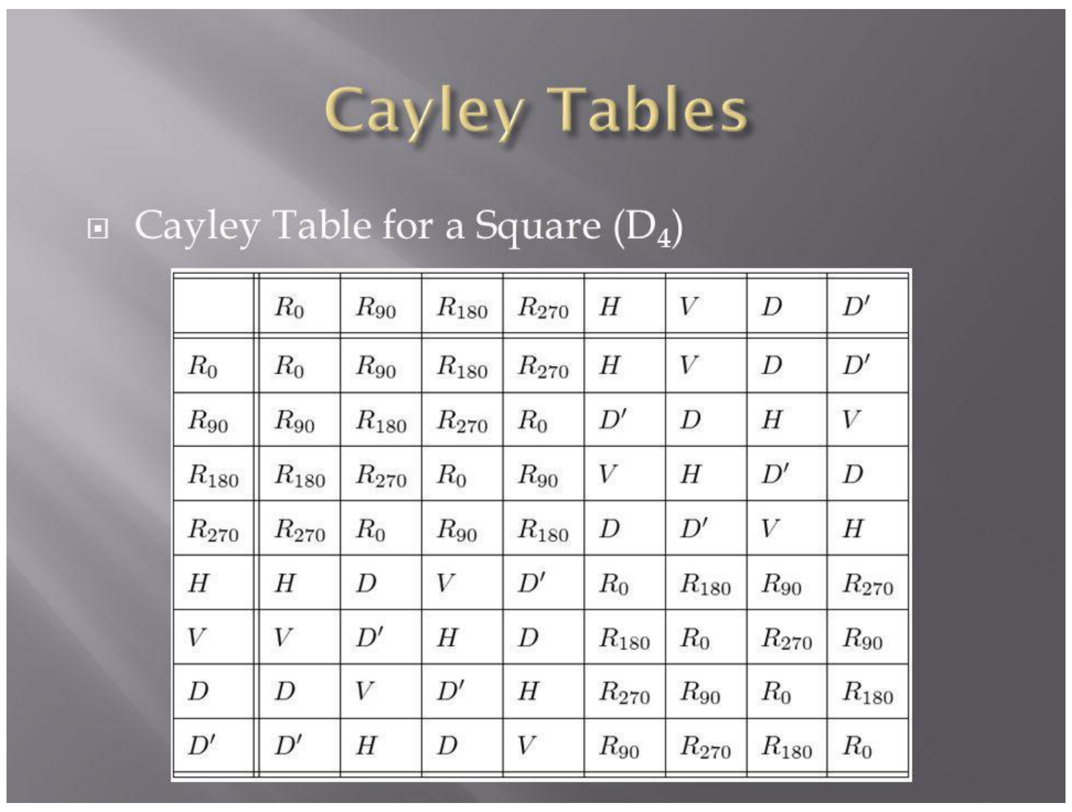
Cayley's definition of a group

Cayley definition of a group can be summarised as a set which obeys the following axioms under an operation $*$.

- Let S be a set and $*$ be a binary operation.
- An identity element denoted by *id* must be in S and for any element a then $a * id = id * a$.
- Let a, b and c be elements of b. $(a * b) * c$ must be equal to $a * (b * c)$ for all elements of S and S must be closed under the operation $*$.
- If a is an element of S then the inverse of a under the operation $*$ must also be an element of S denoted by a^-1 and by definition $a * a^-1 = id$ where id is the identity element for all a in S.

Consequences of Cayley's definition

The definition Cayley gave allowed him and other mathematicians to investigate the theory of groups further. In following papers published by Cayley he introduced graphs and tables later names Cayley graphs and Cayley tables in his honour. Previous to his definition when a mathematician studied a group it was in fact a permutation group. It was later shown that permutation groups were also groups under Cayley's definition. Group theory was arguably the most important part of mathematical research in the 20th century, according to Wikipedia 'One of the most important mathematical achievements of the 20th century was the collaborative effort, taking up more than 10,000 journal pages and mostly published between 1960 and 1980, that culminated in a complete classification of finite simple groups'. None of this research and accomplishments would of been possible without Cayley's definition of the abstract group.



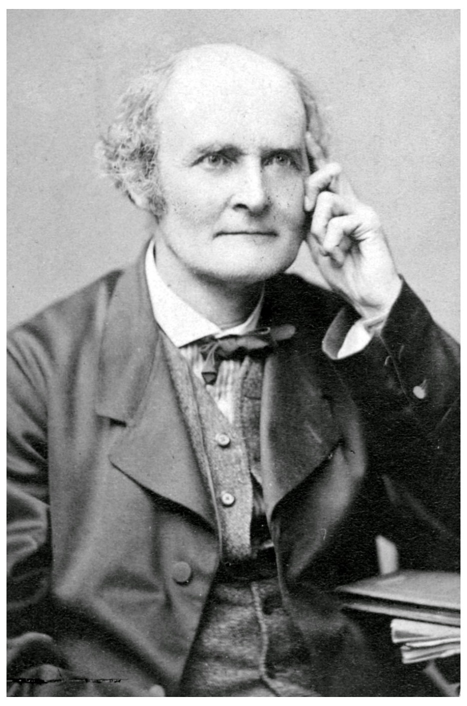
Cayley Tables

Cayley Table for a Square (D_4)

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

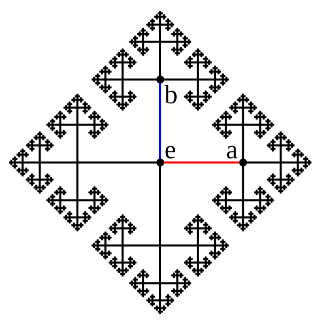
Arthur Cayley (1821-1895)

Arthur Cayley (1821-1895) was a British mathematician who made massive developments to algebra. Cayley gave massive insights into algebra such as the Cayley-Hamilton equation. He was originally a lawyer for 14 years. While investigating the solutions of non-linear equations Cayley proposed the first definition of a mathematical group as we understand them today. Prior to Cayley, the term groups referred to permutation groups.



Below we can see photos of a Cayley table and a Cayley graph

	$\mathbf{e}$	$\mathbf{R_1}$	$\mathbf{R_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$
$\mathbf{e}$	e	R_1	R_2	S_1	S_2	S_3
$\mathbf{R_1}$	R_1	R_2	e	S_3	S_1	S_2
$\mathbf{R_2}$	R_2	e	R_1	S_2	S_3	S_1
$\mathbf{S_1}$	S_1	S_2	S_3	e	R_1	R_2
$\mathbf{S_2}$	S_2	S_3	S_1	R_2	e	R_1
$\mathbf{S_3}$	S_3	S_1	S_2	R_1	R_2	e



Cayley's theorem

Another important contribution to group theory attributed to Cayley is Cayley's theorem. Cayley's theorem unifies the previous notion of permutation groups and Cayleys modern definition of groups. Although Cayley never proved the theorem he showed the 1-to-1 correspondence in his 1854 paper hence it was named after Cayley in his honour.

Theorem (Cayley)

Cayley's theorem states that every group G is isomorphic to a subgroup of the symmetric group acting on G.

Corollary

- *The set of all non-isomorphic finite groups is countable.*
- *Every finite group G of order n is isomorphic to a subgroup of S_n.*

References

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