Representation Theory of Finite Groups

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Introduction

Representation Theory of Finite Groups is a study that dates back to the late 19th century. The original purpose of Representation Theory was to obtain deeper information about finite groups via methods of linear algebra, for example a groups actions on vector spaces may reveal more information about said group. The first scholars to engage in the study of this topic were Frobenius, Schur and Burnside.

What you need to know, the basics!

Definition:
A representation of a group G is a homomorphism $\phi : G \rightarrow GL(F)$ for some finite dimensional non-zero vector space V. The Dimension of V is called the degree of ϕ.

In simpler words, if G is a group and F a field then a matrix representation of G over F of degree 'n' is a homomorphism.

It is a way of assigning to each g ϵ G an invertible $n * n$ matrix ϕ(g) subject to the requirement that ϕ(gh) = ϕ(g)ϕ(h) for all g, h ϵ G

This automatically implies that $\phi(1_g = 1_{GL(V)}$ since group homomorphisms always take identity elements to identify elements.
Also M(g^-1) = M(g)$^-1$

The matrices M(g) are called the Representing Matrices

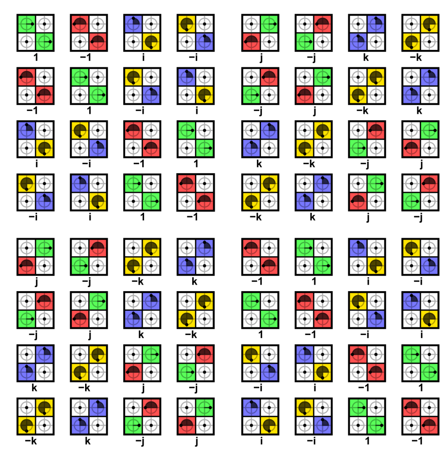
Representation of Quaternion Group

A simple example of representation theory of a finite group by matrices represents the following Quaternion Group; $Q\_8 = e, \bar{e}, i, \bar{i}, j, \bar{j}, k\bar{k} \rightarrow GL\_2(C)$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \bar{e} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \imath = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \bar{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \bar{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \bar{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, k = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

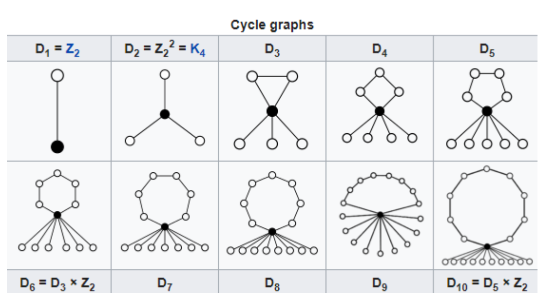
How the numerical and complex values are assigned to the matrices can be seen cleary below in figure 1, and in figure 2 we can see the Cayley table of how the matrices interect when multplied with each other:



×	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
e	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
$\bar{e}$	$\bar{e}$	e	$\bar{i}$	i	$\bar{j}$	j	$\bar{k}$	k
i	i	$\bar{i}$	$\bar{e}$	e	k	$\bar{k}$	$\bar{j}$	j
$\bar{i}$	$\bar{i}$	i	e	$\bar{e}$	$\bar{k}$	k	j	$\bar{j}$
j	j	$\bar{j}$	$\bar{k}$	k	$\bar{e}$	e	i	$\bar{i}$
$\bar{j}$	$\bar{j}$	j	k	$\bar{k}$	e	$\bar{e}$	$\bar{i}$	i
k	k	$\bar{k}$	j	$\bar{j}$	$\bar{i}$	i	$\bar{e}$	e
$\bar{k}$	$\bar{k}$	k	$\bar{j}$	j	i	$\bar{i}$	e	$\bar{e}$

Representation of dihedral groups

Another useful function of Representation Theory is the ability to represent any dihedral groups reflections and rotations.



The general equation to represent a dihedral group of order n is as follows:

$$r_k = \begin{pmatrix} cos(\frac{2.\pi.k}{n}) & -sin(\frac{2.\pi.k}{n}) \\ sin(\frac{2.\pi.k}{n}) & cos(\frac{2.\pi.k}{n}) \end{pmatrix}$$

$$s_k = \begin{pmatrix} cos(\frac{2.\pi.k}{n}) & sin(\frac{2.\pi.k}{n}) \\ sin(\frac{2.\pi.k}{n}) & -cos(\frac{2.\pi.k}{n}) \end{pmatrix}$$

Where r_k expresses anticlockwise rotations through $\frac{2.\pi.k}{n}$ and s_k expresses a reflection across a line that makes an angle of $\frac{2.\pi.k}{n}$ with the x-axis

Note: n refers to the cycle of n vertices and

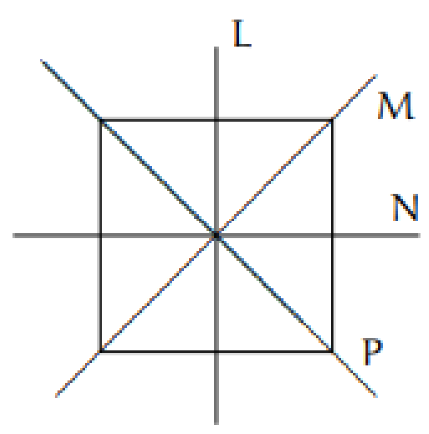
$$k = \sum_{k=0}^{n} n$$

The dihedral group of order 8

Using representation theory it is possible to describe how the symmetries of a square and their multiplications affect each other.

Each rotation and reflection have their own specific matrices, and once multiplied together they give the same results as the refelctions and rotations have on the square itself.

Below is symmetries of a square and their corresponding matrix.



$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} R_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S_N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_P = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Burnside's Theorem

An example of a group-theoretic theorem proved using representation theory is $Burnside's Theorem$ which states that a finite group having no more than two distinct prime divisors must be solvable.

Maps between Representations

$$\begin{array}{ccc} V_\rho & \xrightarrow{T} & V_\tau \\ \downarrow{\scriptstyle \rho(s)} & & \downarrow{\scriptstyle \tau(s)} \\ V_\rho & \xrightarrow{T} & V_\tau \end{array}$$

Theorem

Let V be a representation of G. A subrepresentation is a linear subspace $W \subset V$ such that $gw \in W$ for $g \in G, w \in W$

Definition

If V is a representation such that the only subrepresentations are 0 and V, we say that V is is irreducible

References

https://en.wikipedia.org/wiki/Representation_theory_of_finite_groups
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