Mathematics of Sudoku — Darragh Mac Manus and Cian Nolan

Introduction

A Sudoku is a 9×9 square grid containing 81 cells. The grid is subdivided into nine 3×3 blocks. Some cells are filled in with numbers from the set $\{1, 2, ..., 9\}$. See Fig. 2 (ignore the blue numbers, for now). The goal is to fill in the grid using the nine digits so that each row, each column, and each block contains each number exactly once. We call this constraint on the rows, columns, and blocks the ONE RULE.

Group Theory and Sudoku Puzzles

When considering this topic, a natural question is how many different Sudokus can be created? This number has been computed in [1]. However, you can look at two different Sudoku puzzles and consider them to be essentially the same. For example, in Fig. 2, if you just swapped the numbers 3 and 5, you get a different, valid Sudoku, but you would still solve it the same way, so is it really that different? Similarly, rotating the Sudoku doesn't really make it a new puzzle. All the operations that can be performed on a Sudoku after which it remains a Sudoku are listed below.

1. Relabeling the nine digits
2. Permuting the three stacks
3. Permuting the three bands
4. Permuting the three columns within a stack
5. Permuting the three rows within a band
6. Any reflection or rotation of D_8

A further question is to ask how many "essentially different" sudokus there are, by which, we mean there isn't a sequence of the operations (1)–(6) which transforms one of the sudokus into another. To do this we use the fact that the *operations (2)-(6) is group* of symmetries of the sudoku puzzle. This is like how D_8 is group of symmetries of the square. Here's an outline of the method [2]:

(A) Let 2 sudokus be "equivalent" if they are identical under relabeling i.e under (1).

(B) Consider the group G of symmetries of the set of all sudokus S i.e. operations (2)-(6) *act* on S.

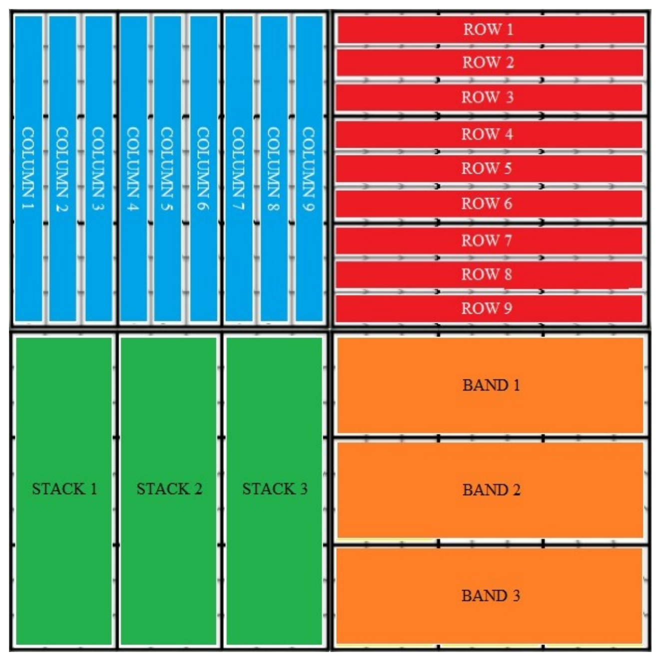
(C) We want to count the number of sudokus we can transform S into which are not "equivalent" i.e using G, how many *essentially different sudokus* can S be transformed into? What we are asking here is how many orbits does S have (under G)?

(D) Burnside's Lemma tells us that No. of Orbits of S under $G = |G|^{-1} \sum_{g \in G} |S^g|$ where S^g is the set of elements in S fixed by g.

(E) It turns out that if 2 elements $g, h \in G$ are in the same conjugacy class, $|S^g| = |S^h|$ i.e. any two elements in the same conjugacy class fix the same number of elements in S. So we need only calculate the number of fixed elements for one element in each class.

(F) There turns out to only be 275 conjugacy classes. [ref] used computer software GAP to see that only 27 classes have fixed elements and brute-forced how many there are. It only remains to divide this number by $|G|$ to find the number of orbits of S under G, that is 5,472,730,538. (Note: $|G|$ is found by calculating the number of elements in operations (2)-(6)).

Fig. 1. [3]



Solving a Sudoku: Basic Strategies

When solving a Sudoku by hand, common practice is to scribble a small note of the numbers which could be entered into that cell without breaking the ONE RULE (We'll call these cell candidates). An example of this is shown in Fig 2. In cell A1, 2,5,7 and 8 are written in blue because the numbers 1,3 and 4 already exist on column A and the numbers 6 and 9 exist on row 1 (3,4,6 and 9 appearing in the block ABC123 would also prevent these appearing as candidates). Usually, writing all possibilities for all cells is unnecessary, but, for the purposes of explaining the following techniques, we'll include these 'helper' notes.

1. Naked singles

Look at the sudoku puzzle with its candidates filled in. If there is a cell with a sole value available to to be placed, the cell has this value. For example, in Fig. 2, 8 must be placed in A5.

2. Hidden Singles

If a cell is the only one in a row, column or block that can take that particular value, then it must have that value. This can be seen in Fig. 2, in cell C9. C9 is the only cell in its block that can take 6 as a value so C9 must be 6. This argument (and others) apply to rows, columns and blocks in the same way, so that sometimes one considers the three objects (i.e. rows, columns & blocks) to all be 'virtual lines'.

Fig. 2.

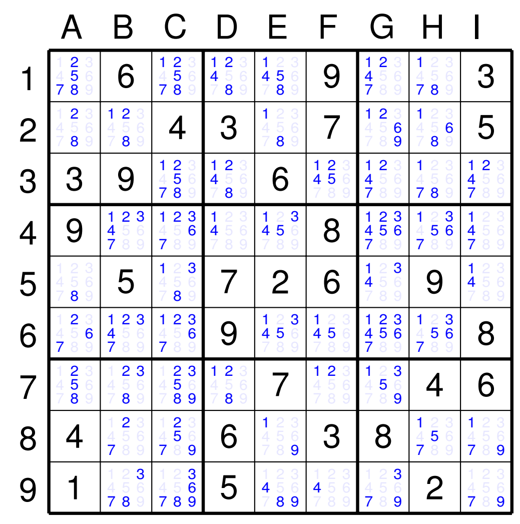


Fig. 3.

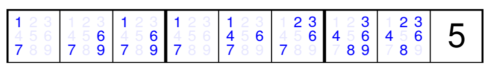


Fig. 4.

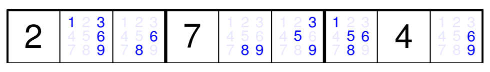
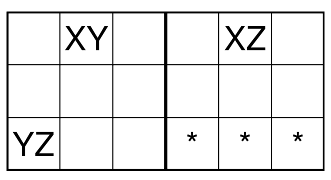


Fig. 5.



Solving a sudoku: Additional Strategies

So far, all strategies have followed directly from the ONE RULE and in practice, writing all those 'helper' notes are unnecessary. Here, however, it may be useful.

3. Locked candidates

Sometimes, there is a block where the only possible positions for a number, say 7, are in the same row (or column). That eliminates the possibility of 7 appearing anywhere else in that row (or column). In Fig. 2, look at D7 and F7. They are the only possible cells in the block DEF789 with 2 as candidates. So, 2 *must* appear in either D7 or F7. Therefore, 2 can't appear anywhere else in row 7. This eliminates 2 as a candidate in A7,B7 and C7.

It should be clear how this idea is used for eliminating candidates in blocks, instead of rows or columns. Look at column I in Fig. 2. See if you can figure out how to use this column to eliminate candidates from block GHI789.

4. Naked pairs, triplets, quads,...

These are similar to naked singles except, now you have two candidates in two cells (or replace three with some positive integer for naked triplets, quads, etc.).

Say you have two cells on the same "virtual line" (row, column or block) which have exactly the same two possible candidate numbers. Then those two numbers are eliminated as possibilities elsewhere on that virtual line. This occurs in Fig. 3. which is a row of some sudoku. The first and fourth cells contain exactly 1 and 7, so 1 and 7 must appear in those cells in some order. This eliminates 1 and 7 as candidates elsewhere on the row.

With naked pairs, both cells must have exactly the same two candidates, but for naked triplets, quads, etc., the only requirement is that the three values be the only values appearing in those cells on some virtual line. Fig. 4 has a naked triplet with the numbers 6,8 and 9. Can you see it?

5. Hidden pairs, triplets, quads,...

Hidden pairs, triplets and quads, etc. are the natural extensions of hidden singles. If there are two cells with two candidates between them that don't appear elsewhere in the same virtual line, then any other candidates for those cells can be eliminated. Fig 4 has a hidden triplet with the numbers 1,3 and 5.

Solving a Sudoku: Complex Strategies

6. X-Wings and Swordfish

An X-Wing configuration occurs when the same candidate occurs exactly twice in two rows and these four cells lie on only two columns (or similarly, if you swap the words "columns" and "rows"). This is illustrated in Fig. 6. The highlighted cells show where 3 occurs exactly twice in rows 3 and 8 and the four cells lie on columns B and G. 3 must occur on either B8 or G8. (i) Say 3 occurs on B8. This eliminates 3 as a candidate for B3 and G8, turning G3 into a hidden single i.e. G3 = 3. (ii) Say 3 occurs on G8. A similar argument shows that B3 = 3. So in either case, 3 is eliminated as a candidate from rows 3 and 8 and columns B and G.

A Swordfish is an extension of an X-wing. Now there are three rows/columns with the three candidates appearing in at most three other columns/rows. There is no requirement for the candidates to appear in all six cells; only two cells are necessary.

7. XY-Wing

The XY-wing gets its name from the letters used in instructive examples rather than Star Wars or marine life. The basic idea is this: Sometimes a cell has only two possible candidates. If we assume one candidate is true, that forces a certain conclusion. If, by assuming the other candidate is true, the same conclusion is forced, the conclusion must be true. Look at Fig. 5. The letters X,Y and Z represent the *sole* candidates for the cells, for example, the XY cell means that by the ONE RULE, X and Y are the only allowable numbers. (i) Assume the XY cell takes Y as its value. Then the YZ cell must take Z as a value eliminating the possibility of the three cells marked with an asterisk. (ii) Assume the XY cell takes X as its value and it's clear that Z is still eliminated as a candidate in the asterisk-marked cells. So the cells marked with an asterisk must not contain Z. There is a second XY-Wing in this configuration. Can you find it?

8. Simple Colouring/Chains

Consider Fig. 7. Begin at cell C8. (i) If C8 takes 1 as its value, then cell D8 cannot be 1 as they share a row (or virtual line). (ii) If C8 doesn't take 1 as its value, C4 must be 1 as it's the only other legal position (by it being a hidden single) to place a 1 on row 8. We represent this relation "if C8 isn't 1, then C4 is 1" by marking them different signs (+ or -) on the diagram. In this case C8 is a '-' and C4 is '+'. Next we notice that "if C8 isn't 1, then A5 is 1", (using hidden singles again) and we mark the cells different signs, once again. We continue in fashion until we reach the point where we mark D5 as a '+'. This means "D5 isn't 1", meaning D8 cannot be 1. (i) and (ii) both arrive at the same conclusion and since either (i) is true and (ii) is false, or vice-versa, the mutual conclusion is true i.e. D8 cannot be 1. (i) and (ii) were chain-like arguments (although (i) was a very short chain) giving the technique its name.

Fig. 6.

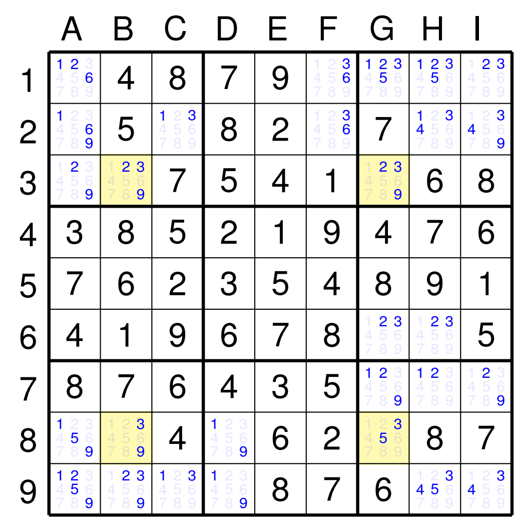
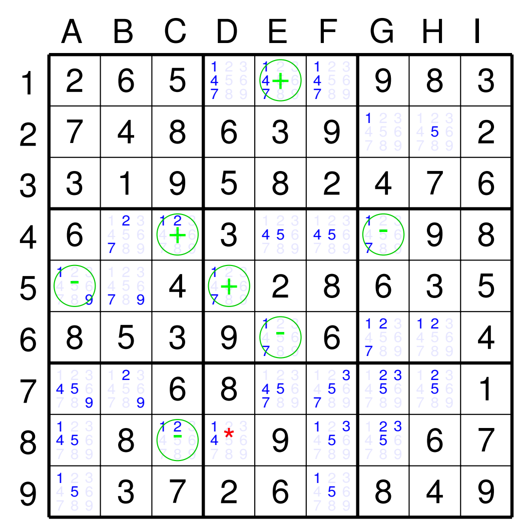


Fig. 7.



References

[1] Felgenhauer, Bertram, and Frazer Jarvis. "Mathematics of Sudoku I." (2006) 17 Aug 2009 http://www.afjarvis.staff.shef.ac.uk/

[2] Russell, Ed, and Frazer Jarvis. "Mathematics of Sudoku II." (2006) 17 Aug 2009 http://www.afjarvis.staff.shef.ac.uk/

[3] http://this-is-sudoku.blogspot.com/2013/05/conventions-used-in-sudoku.html

[4] http://www.sadmansoftware.com/sudoku/solvingtechniques.php

[5] Tom Davis. *Expeditions in Mathematics* (2011) MAA, pp. 31-60