Frieze Groups

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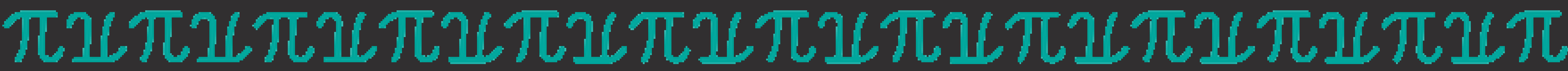


Introduction

Our chosen topic is frieze groups. A **frieze group** is an infinite discrete group of symmetries, i.e. the set of the geometric transformations that are composed of rigid motions and reflections that preserve the pattern. A **frieze pattern** is a two-dimensional design that repeats in one direction. A study on these groups was published in 1969 by **Coxeter** and **Conway**. Although they had been studied by Gauss prior to this, Gauss did not publish his studies. Instead, they were found in his private notes years after his death.

Visual Perspective

There are seven types of groups. Each of these are the symmetry group of a frieze pattern. They have been illustrated below using the Conway notation and corresponding diagrams:

- The first, and the one which must occur in all others is Translation, or a 'hop'.

- The second is a glide with translation, or a 'step'.

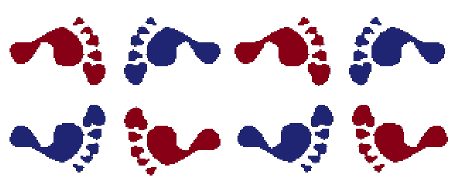
- The third is a translation with a vertical reflection, or a 'slide'.

- The fourth is a translation with a 180° rotation, or a 'spinning hop'.

- Fifth being a translation with vertical reflection, glide reflection and 180° rotation, or a 'spinning slide'.

- Sixth being translation with horizontal glide reflections, or a 'jump'.

- The last is a horizontal and vertical reflection with translation 180°, or a 'spinning jump'.



- The pattern formed above using π is a frieze pattern! It is a translation with a 180° rotation.

Numerical Perspective

A frieze pattern is an array of natural numbers, displayed in a lattice such that the top and bottom lines are composed only of 1's and for each unit diamond,

$$\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \qquad \text{the rule } (bc) - (ad) = 1 \text{ holds.}$$

Here is an example of a frieze pattern:

... 1 1 1 1 1 1 1 1 1 ...
... 1 2 3 1 3 1 4 1 ...
... 3 1 5 2 2 2 3 3 1 ...
... 2 2 3 3 1 5 2 2 ...
... 1 3 1 4 1 2 3 1 3 ...
... 1 1 1 1 1 1 1 1 ...

[2]

Properties

- Every pattern of order n (i.e. with n lines) admits an horizontal translation of length n + 1.
- Every pattern of order n is completely determined by a sequence of n + 1 numbers.
- Every pattern is completely determined by the numbers on the second line. (★)

References

Y. Liu and R.T. Collins. "Frieze and Wallpaper Symmetry Groups Classification under Affine and Perspective Distortion". In: (1999).

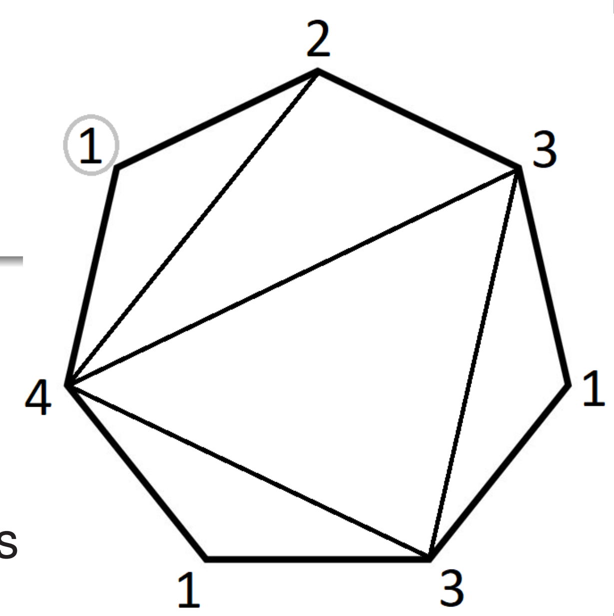
J.F. Marceau. "Frieze Patterns and Triangulated Polygons". In: (2013).

What second-row numbers validate a frieze pattern? (★)

The second-row numbers that determine the pattern are not random. Instead, these numbers have a peculiar property.

Theorem

There is a bijection between the valid sequences for frieze patterns and the number of triangles adjacent to the vertices of a triangulated polygon. [2]



The triangulated polygon on the right determines the lattice example from earlier. It has 7 sides, which indicate the period of the frieze pattern. Note that the adjacent triangles to each vertex matches the second-row numbers, starting at ①.

Applications

Frieze patterns are used very often in **decorative art** and **architecture** worldwide and for centuries, before frieze group theory even surfaced. Friezes in architecture are long stretches of painted, sculpted or calligraphic decorations, sometimes depicting scenes in a sequence of discrete panels. They are used most famously in Greek and Roman architecture.

Frieze groups have been used to develop an algorithm to detect repeating patterns in images. This can be used for **object recognition** in images, even where the pattern is distorted by perspective [1].