The History of Lagrange's Theorem

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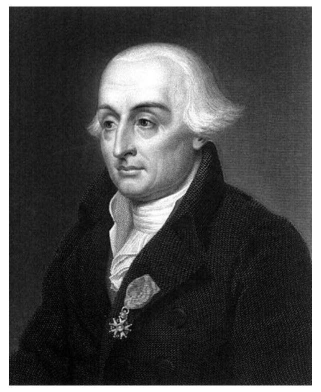
Introduction

The Italian mathematician Joseph Louis Lagrange created a very famous theorem in groups and applied mathematics known as Lagrange's Theorem. In this poster you will learn the history and evolution of this theorem from Lagrange's original version to the modern adaptation of the theorem. We will break this topic into different sections featuring details on Lagrange's life, his contributions to mathematics and the differences between his original theorem and the one we see today.

Who was Joseph Lagrange?



Joseph Luis Lagrange (Giuseppe Luigi Lagrangia) was born in Turin, Italy on the 25th of January 1736 and lived until 1813. He is well known because of his contributions in many fields of mathematics such as number theory, number analysis and applied mathematics. An example of one of his contributions is Lagrangian mechanics, which re-formulated Newtonian mechanics to simplify formulae and calculations.

Lagrange's Theorem

Trying to find the subgroups of some finite group G could prove very difficult. Lagrange's Theorem makes finding those subgroups much easier.

Theorem

If G is a finite group and H is a subgroup of G, then $|H|$, the order of H divides $|G|$, the order of G.

This proves very helpful in figuring out which subgroups a group possesses provided the group is **finite**.

Solving the Quintic

Solving linear, quadratic, cubic and quartic equations can be done by finding and factorizing their radicals, no matter whether the roots are rational or irrational, real or complex. There should always be a formula that provides the desired solutions. However, there is no algebraic expression in terms of roots for the solutions of general quintic equations. This statement is known as the **Abel–Ruffini theorem**([1]), which was first asserted in 1799 and completely proved in 1824. This result also holds for equations of higher degrees. An example of a quintic whose roots cannot be expressed in terms of radicals is

$$x^5 - x + 1 = 0.$$

This particular quintic is in **Bring–Jerrard normal form**. Some quintics may be solved in terms of radicals. However, the solution is generally too complex to be used in practice. Instead, numerical approximations are calculated using a root-finding algorithm for polynomials, for example, Newton's Method.

What did Lagrange do?

His concern was the question of finding an algebraic formula for the roots of the general 5th degree polynomial and more generally for the nth $(n > 4)$, since the quadratic, cubic and quartic formulae were already known. He observed that the solutions to quartic and cubic equations involved solving supplementary polynomials of lower degree. These polynomials are also known as **"resolvent"**([2]) polynomials. For this example we can write the roots as:

$$\frac{x_1x_2 + x_3x_4}{2}, \frac{x_1x_3 + x_2x_4}{2}, \frac{x_1x_4 + x_2x_3}{2}$$

where x_1, x_2, x_3, x_4 are roots of the original polynomial. He also observed that all four roots could be permuted in $4! = 24$ possible ways and only these three values would occur. This lead him to say: To solve 5th degree polynomials one should try to find function in 5 variables that takes on 3(or 4) different typical values when the variables are permuted in all $5!$ ways. He was unable to determine if such a function existed, but he did come up with, in essence, the following theorem.

Lagrange's Original Theorem

As you can see below, Lagrange's Original Theorem greatly differs from the theorem we see today as it involves changing functions by permuting their values.

Theorem

If a function $F(x_1, x_2, \ldots, x_n)$ of n variables is acted on by all $n!$ possible permutations of the variables and these permuted functions take on only r distinct values then r is a division of $n!$.

Example: Lagrange's Theorem for C_6

We can use C_6 as an example to show the main ideas behind Lagrange's Theorem. $C_6 = \{1, g, g^2, g^3, g^4, g^5\}$ (where $g^6 = 1$) has as one of its subgroups $H = \{1, g^3\}$. If we multiply H on the right by each element of C_6 in turn we find the different right cosets of H in G.

$$\{1, g^3\}, \{g, g^4\}, \{g^2, g^5\}$$

If we use three different colours for each of the right cosets we see that $H = \{1, g^3\}$ shifts through each element of C_6 as we multiply H on the right by elements of C_6.

$$C_6 = \{1, g, g^2, g^3, g^4, g^5\}$$

There are **three** distinct cosets.

- Two cosets are either equal or disjoint.
- Every element of G is in exactly one right coset.
- Each right coset is the same size as H.

These are the key ideas needed to prove Lagrange's Theorem.

References

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