The Group of Symmetries of the Cube

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Introduction

The cube is a shape made up of 6 faces, 8 vertices and 12 edges. The cube has 24 rotational symmetries and they form a group that is a copy of S4, the group of permutations of four objects. It also has 24 reflections among the 9 planes of reflection. This makes up in total the 48 isometries of the cube.

All 24 permutations of the diagonals can be generated by any 2 rotations

Let the rotation A be a rotation clockwise about the vertical axis (as shown in the diagram) and Rotation B be a rotation about the horizontal axis.

- Rotation A (R_A)-
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix}$$
i.e the rotation takes 1 -> 2 for example

- Rotation B (R_B)-
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 7 & 3 & 1 & 5 & 8 & 4 \end{pmatrix}$$
i.e the rotation takes 3 -> 7 for example

Now we can combine both of these matricies of permutations to generate another, and/or combine it with itself multiple times to get a get different permutations. For Example,

- $(R_A)^2 =$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix}$$
Notice here that 1 -> 2 -> 3, i.e 1 -> 3
  - $(R_A)^3$ also produces a new permutation, i.e sending 3 -> 4 -> 1 -> 2, i.e 3 -> 2.
  - Same goes for $(R_B)^2$,$(R_B)^3$, But when you apply a rotation a 4th time, i.e $(R_A)^4$ or $(R_B)^4$, each vertex returns to its original position, also know as the identity rotation (I).

- We can now induce a subgroup of order 8 =
$\{ I , R_A , (R_A)^2 , (R_A)^3 , (R_B)^2 , (R_B)^2 R_A , (R_B)^2 (R_A)^2 , (R_B)^2 (R_A)^3 \}$

- And a subgroup of order 3
$\{ I , R_A R_B , (R_A R_B)^2 \}$

- Calculating the $LowestCommonMultiple$ (lcm) of both orders of the subgroups give you 24, i.e the number of all permutations generated by 2 rotations.

Graphical Visualisation

rotation.png rotation.png

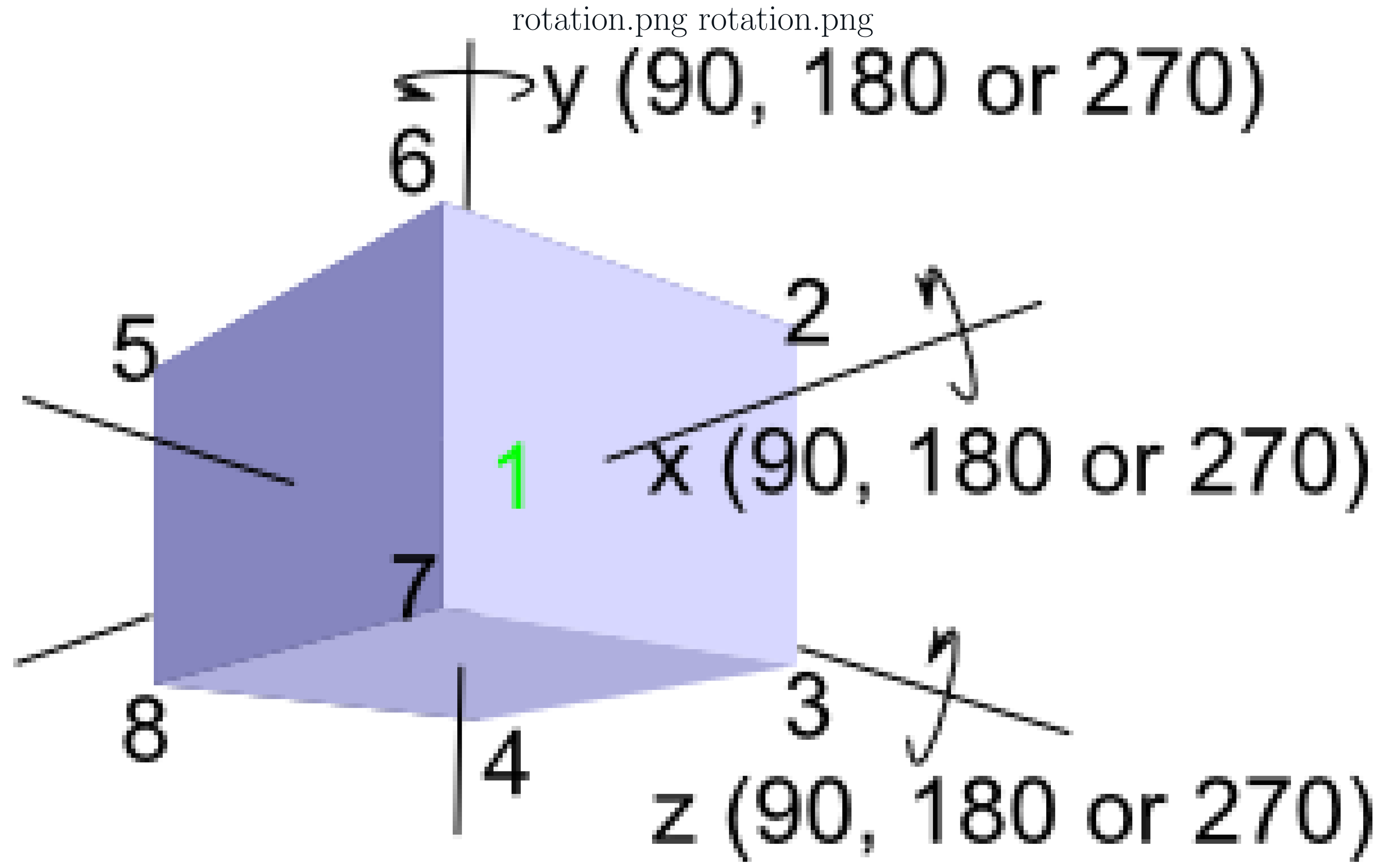

Fig. 1: The Rotations of The Cube

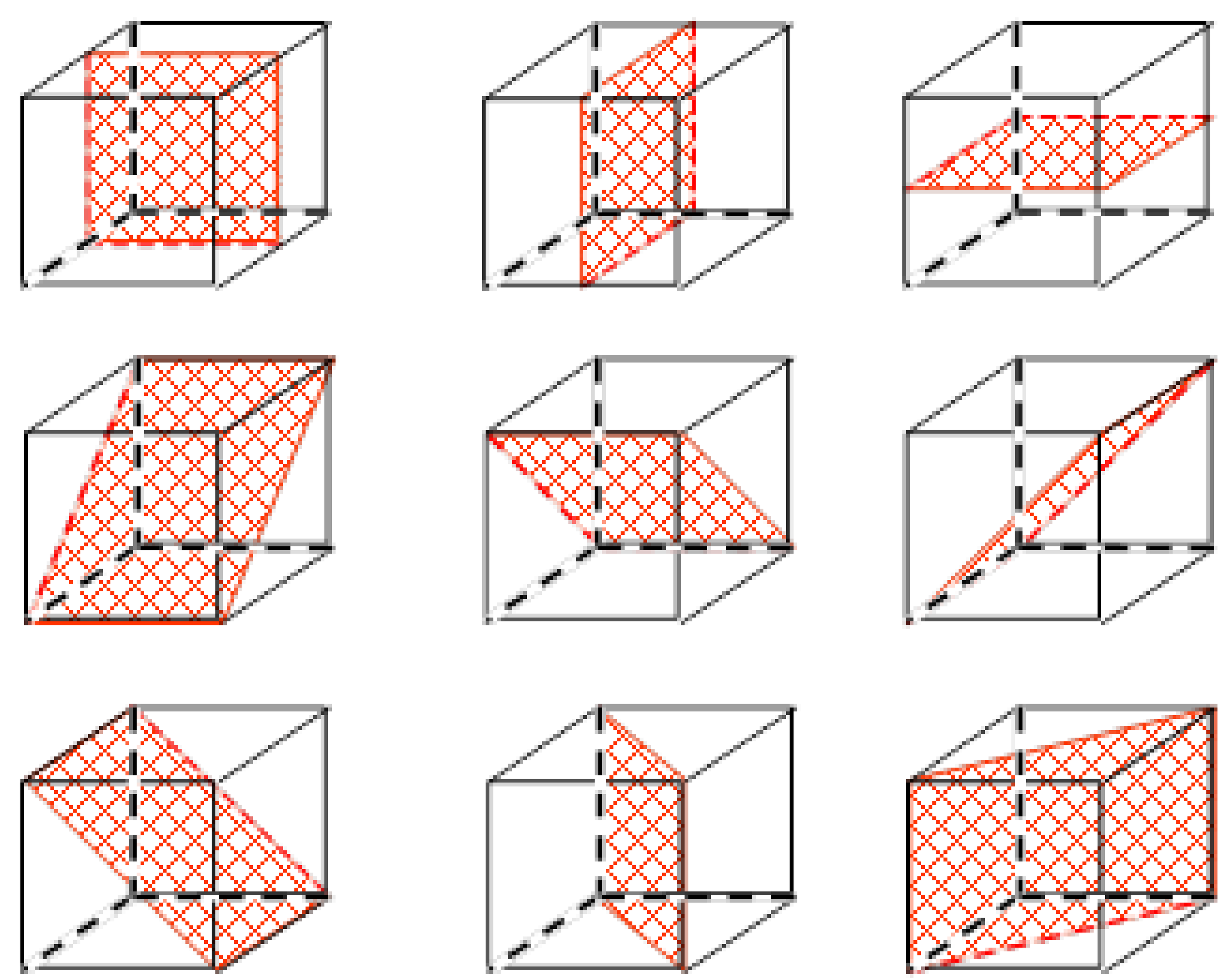

Fig. 2: The Reflection Planes of The Cube

Rotational Matrices

Take a set of all 3x3 permutation matrices and assign a + or a - to each of the 1's in the matrix. There are 6 matrices and 8 different sign permutations, that's 48 matrices in total giving the whole group of symmetries of the cube. There are exactly 24 matrices with determinant = +1 and these are the rotational matrices of the octahedral group.

$$\begin{vmatrix} +1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ +1 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

The other 24 matrices with determinant of -1 correspond to a reflection or inversion.

$$\begin{vmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$\begin{vmatrix} +1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{vmatrix}$$

Orbit Stabilizer Theorem

Let G be a group that acts on a set X, let x be an element of X, let O_x be the orbit of x and let S_x be the stabilizer of x. Then $|O_x||S_x| = |G|$.
In words, the size of the group is the size of the orbit times the size of the stabilizer.
Proof: Here are two equivalence relations on G. For the first, I define $g \sim_1 h$ if $gx = hx$. For the second, I define $g \sim_2 h$ if $h^{-1}g \in S_x$. Note that this is the same as saying that $g^{-1}h \in S_x$ and also the same as saying that $gS_x = hS_x$.
Now $gx = hx$ if and only if $h^{-1}gx = x$ if and only if $h^{-1}g \in S_x$. So the two equivalence relations are the same. The number of equivalence classes for the first relation is $|O_x|$ and the size of each equivalence class for the second relation is obviously $|S_x|$,so the result is proved.

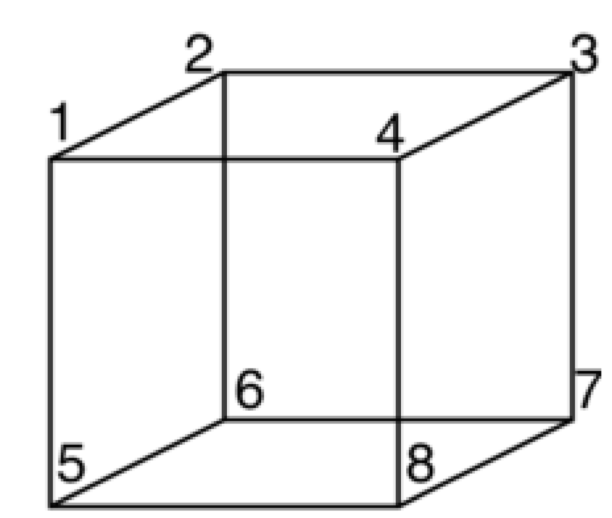

Fig. 3: Points of Orbit of the Cube