Lagrange's Theorem

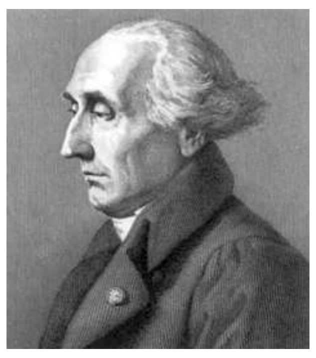
Caroline Wilson

Introduction

Lagrange's Theorem is a famous theorem in Group Theory and takes it's name from the Italian mathematician Joseph Louis Lagrange who lived from 1736 to 1813. In simple language this theorem says that if H is a subgroup of a finite group G then the size of H divides the size of G. But the original statement of the theorem came before the modern definition of a group see [1]. In this poster we will explore the origins and links between the original ideas around Lagrange's Theorem and it's modern statement today.

Joesph Louis Lagrange

Giuseppe Luigi Lagrangia was born in Turin on the 25th January 1736. He made contributions to many areas of mathematics including analysis, number theory and mechanics. His name is associated with topics such as the Lagrangian in mechanics and the Euler-Lagrange equation in calculus.



Lagrange's Theorem

A natural question to ask when studying a finite group G is "what subgroups does G have?" This is often a very difficult question to answer but Lagrange's Theorem offers some knowledge about the answer to the question.

Theorem

If G is a finite group and H is a subgroup of G, then the order of H divides the order of G.

This theorem goes some way towards helping us decide which subgroups a particular group might have. For example, it tells us that a group G of order 24 cannot possibly have a subgroup of order 11.

Insolvability of the Quintic

The solutions of any quadratic polynomial $ax^2 + bx + c = 0$ can be expressed in terms of the coefficients a, b and c using addition, subtraction, multiplication, division and square roots via the familiar quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Similar formulae for cubic and quartic polynomial equations using square roots and cube roots have been known since the 16th century. However, the search for a similar formula to solve the quintic equations proved less fruitful for mathematicians. In 1824 the Abel-Ruffini Theorem was proved and it asserted that there is no such formula to solve general quintic equations. The intervening centuries saw mathematicians work furiously on solving this problem. Along the way many rich and exciting mathematical theories were developed.

What did Lagrange do?

Lagrange noticed that to solve a cubic equation one could study an equation of degree two which was related to the original, he called this lower degree equation a "resolvent" ([1]). For example "the quartic was solved using a cubic resolvent polynomial whose roots could be written as

$$\frac{x_1x_2 + x_3x_4}{2}, \frac{x_1x_3 + x_2x_4}{2}, \frac{x_1x_4 + x_2x_3}{2}$$

where x_1, x_2, x_3, x_4 were the roots of the original polynomial." ([1]) Furthermore, he noticed that if the four roots x_1, x_2, x_3, x_4 are permuted in all possible ways $4! = 24$ only these three values would occur. Today we know that the symmetric group on four points S_4 is the group or all permutations of $\{1, 2, 3, 4\}$ and that this group has $4! = 24$ elements. Lagrange postulated that to solve a quintic one would need to find a function which can only take on four values when the variables are [permuted in $5! = 120$ ways.

Theorem (Lagrange's Original Theorem)

If a function $F(x_1, x_2, \ldots, x_n)$ of n variables is acted on by all $n!$ possible permutations of the variables and these permuted functions take on only r distinct values then r is a division of $n!$.

Lagrange's Theorem for C_8

We can illustrate the key ideas behind the proof of Lagrange's Theorem using the example of $C_8 = \{1, g, g^2, g^3, g^4, g^5, g^6, g^7\}$ (where $g^8 = 1$) which has as one of its subgroups $H = \{1, g^4\}$. If we multiply H on the right by each element of C_8 in turn we find the different right cosets of H in G.

$$\{1, g^4\}, \{g, g^5\}, \{g^2, g^6\}, \{g^3, g^7\}$$

Using some color we can see how $H = \{1, g^4\}$ gets "shifted" through the elements of C_8 as we multiply H on the right by elements of C_8.

$$C_8 = \{1, g, g^2, g^3, g^4, g^5, g^6, g^7\}$$

There are **four** distinct cosets.

- Each right coset has the same size as H.
- Two cosets are either equal or disjoint.
- Every element of G is in exactly one right coset.

These are the key ideas needed to prove Lagrange's Theorem.

References

Richard L. Roth. "A History of Lagrange's Theorem on Groups". In: *Mathematics Magazine* 74.2 (2001), pp. 99–108. ISSN: 0025570X, 19300980. URL: http://www.jstor.org/stable/2690624.