Formulation of the Group Concept

Financial Mathematics and Economics, MA3343 Groups

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Abstract

This study consists on the **group concepts** and the **four axioms** that form the structure of the **group**. Provided below are examples to show different mathematical operations within the **elements** and how this form the different characteristics of the **group**. Furthermore, a detailed study of the **formulation of the group theory** is performed where three major areas of modern group theory are discussed:**The Number theory, Permutation** and **algebraic equations theory** and **geometry**.

What are Group

A **group** is an **algebraic structure** consisting of different **elements** which still holds some common features. It is also known as a set. All the **elements** of a **group** are equipped with an algebraic operation such as addition or multiplication. When two **elements** of a **group** are combined under an operation, they form a third element of the group with the same features.
An **example** of a group: Set of Integers: It's usually represented with "Z " and it consists of all the numerical numbers as below:
$Z = \{0, \pm1, \pm2, \pm3, \ldots\}$

Structure of the Group

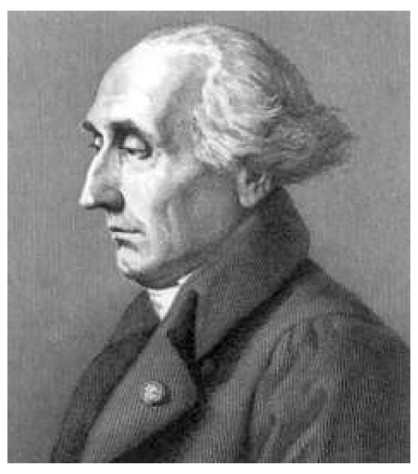
Group Axioms:
All the **elements** of a **group** satisfy four characteristics known as **group axioms**. These include: closure, associativity, Identity and invertibility.
Closure If an algebraic operation is performed on the elements of a group and the result is always a member of the group, then the group is equipped with the closure property.
Associativity: If the order of algebraic operations does not alter the sequence of the elements of a group, the group possesses the characteristic of associativity.
Identity: If any binary operation does not alter the features of an element in a group, the group is equipped with the identity characteristic.
Invertibility: If an element can negate the effect of combination with another element, the group possesses the characteristic of invertibility. The inverse operation on an element results in identity.

References

- J J O'Connor and E F Robertson. *The development of group theory*. http://mathshistory.st-andrews.ac.uk/HistTopics/Development_group_theory.html. [Online; accessed November 23, 2018]. 1996, May.
- Wikipedia. *History of group theory*. https://en.wikipedia.org/wiki/History_of_group_theory#Late_19th_century. [Online; accessed November 23, 2018]. 2019, January 6.

Fathers of Group Theory

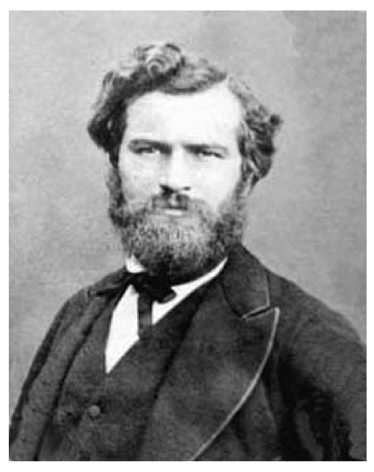

Otto Hölder


Joseph Louis Lagrange


Niels Henrik Abel


Évariste Galois

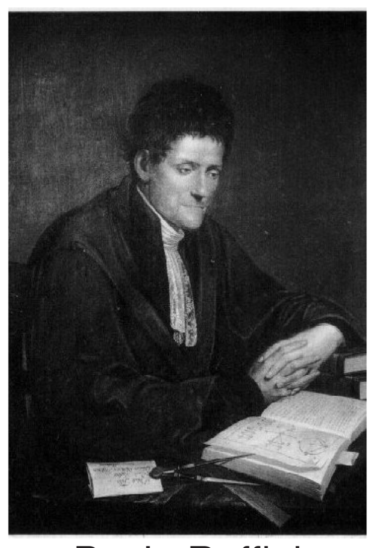

Felix Klein


Camille Jordan

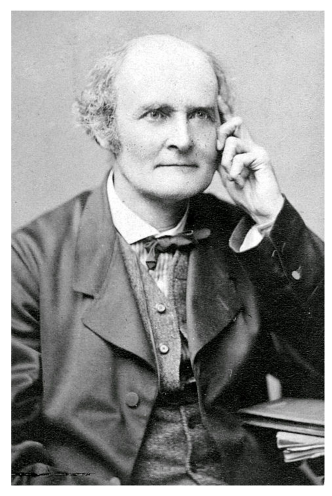

Carl Friedrich Gauss


Leonhard Euler


Ernst Christian Schering


Paolo Ruffini


Augustin-Louis Cauchy


Arthur Cayley

Formulation of the group concept

The **group theory** is considered as an abstraction of ideas common to three major areas: **Number theory, Algebraic equations theory** and **permutations**. In 1761, a Swiss mathematician and physicist, **Leonhard Euler**, looked at the remainders of power of a number modulo "n". His work is considered as an example of the breakdown of an **abelian group** into co-sets of a subgroup. He also worked in the formulation of the divisor of the order of the group. In 1801, **Carl Friedrich Gauss**, a German mathematician took Euler's work further and contributed in the formulating the **theory of abelian groups**. He also examined the orders of elements and proposed the concept of subgroup for every number dividing the order of a **cyclic group**. He worked on the binary quadratic forms as well, i.e.,

$$ax^2 + 2bxy + cy^2$$

Where;
a,b, and c are integers.
Later on in 1869, another German Mathematician, **Julius Schering**, editied **Gauss's** work and founded the basis for the **abelian group**. In 1770, **Joseph-Louis Lagrange**, who was one of the greatest mathematicians and was known for studying algebraic equations (**cubic** and **quadratic**) was the first one who approached **permutations** in general. **Paulo Ruffini**, an Italian mathematician and philosopher, who also demonstrated the insolubility of the equations of degree 5, took ideas proposed by Lagrange and introduced the new concept called **groups of permutations** (which classifies into two types known as **cyclic groups** and **non-cyclic groups**). He also proved that the **associative law** always holds for **permutations**. Later on, a French mathematician, **Louis Cauchy** who played a significant role in developing the **theory of permutations**, introduced concepts related to the **notation of powers, orders of a permutation**, as well as **positive and negative permutations**. He also introduced cycle notation and gave the concept of **identity permutation** with power 0. Next in 1824, **Niels Henrik Abel**, a Norwegian mathematician who found **commutativity** of the **group of a polynomial**, proved the insolubility of the degree 5 equations, where he used the existing ideas of **permutations of roots**. Then in 1832 another French mathematician, **Evariste Galois**, who worked on the **algebraic solutions** of equations related to the **group of permutations**, proposed the concepts of normal and special subgroups. **Galois** was first who fully understood **group theory** structure. He took **Abel's** calculations and putted them into **group theory** context. According to him, a group could be broken down into **co-sets** of a subgroup if the left and right **co-set** coincide. Later, **Camille Jordan** showed the importance of groups permutation and gave the concept of **isomorphism of permutations groups** and proved the **Jordan-Holder theorem** as well. Then, English mathematician, **Arthur Cayley**, extended Cauchy's work on **permutations** and wrote two papers on **abstract group** in 1854. Although it was new area, he defined an **abstract group** and provided a table to display the group multiplication. Besides this, he defined **matrices** and **quaternions** as **groups** for the first time. Afterwards, **Otto Ludwig Holder** further investigated **Cayley's** work and proposed the concept of the groups of order:

$$p^3, pq^2, pqr, \text{ and } p^4$$

In a meantime in 1872, **Felix Klein**, a German mathematician published the Erlangen Program about the group theoretic classification of geometry. Then as far as abstract concept was concerned, **Anton von Dyck**, a German mathematician introduced **free groups** and defined **abstract groups** in terms of relations and generators in 1882 and 1883.