MA3343: GROUPS
SEMESTER 1 2021-22
PROBLEM SHEET 1
Due date (for questions marked with $*$): Friday October 15 2021

Notes: Please answer as clearly and fully as you can, and think about clarity and accuracy as well as about technical details. The numbers appearing in parentheses beside each problem explicitly link the problem to the learning outcomes for the course. [1] Please submit your work by 5.00pm on Friday October 15, as a pdf file via the Blackboard assignment in the "Assignments" section of the MA3343 Blackboard page. Comments and outline solutions will be posted on the website later. Please do not include solutions to problems that are not assigned as homework - these are intended for independent study.

1. **(1,2)** Determine whether each of the following sets is a group. If your answer is that the object *is* a group, it is sufficient to just say so. If not, you should give a reason why not.
   (a) The set of permutations of the set $\{a, b, c, d\}$ that send a to a.
   (b) The set of permutations of the set $\{a, b, c, d\}$ that send a to b.
   (c) $*$ The set of 2×2 matrices with entries in $\mathbb{Z}$ and non-zero determinant, under matrix multiplication.
   (d) The set of *symmetric* matrices in $\mathrm{GL}(2, \mathbb{Q})$, under matrix multiplication (recall that a matrix A is symmetric if $A = A^T$, where A^T is the transpose of A).
   (e) $*$ The set of *orthogonal* matrices in $\mathrm{GL}(2, \mathbb{Q})$, under matrix multiplication (recall that a matrix A is orthogonal if $AA^T = I$, where I is the identity matrix).

   Note: What has to be checked here (and in Question 4 below) is that the set is closed under the operation, that the operation is associative, that the set contains an identity element for the operation and that every element in the set has an inverse in the set, for the operation. As soon as any of these fails, you can report that it's not a group and note a reason.

2. **(2,3,5)** Give an example of
   (a) an infinite non-abelian group.
   (b) an infinite abelian group.
   (c) An abelian group with exactly 10 elements.
   (d) A non-abelian group with exactly 10 elements.
   (e) $*$ A group with four elements, in which every element is its own inverse.
   (f) $*$ A group with four elements, in which not every element is its own inverse.

   Note: Coming up with examples of groups with specified properties takes practice. If in doubt, use the examples of Section 1.1 as a starting point and see if you can adapt them.

3. **(1,2,5)** $*$ Give an example of
   (a) a binary operation on $\mathbb{Z}$ that is both associative and commutative;
   (b) a binary operation on $\mathbb{Z}$ that is neither associative nor commutative;
   (c) a binary operation on $\mathbb{Z}$ that is associative but not commutative;
   (d) a binary operation on $\mathbb{Z}$ that is commutative but not associative.

[1] LEARNING OUTCOMES
By the end of this course you will be able to :

1. Explain what a group is and use the definition of a group to identify examples and non-examples.
2. Use the language and terminology of group theory in an accurate and knowledgeable way.
3. Give examples of groups with certain specified properties.
4. State and prove some major theorems of group theory.
5. Identify and discuss important features of finite groups.
6. Critically assess proposed proofs of statements in group theory, and write some proofs of your own.

Note: The point here is to recognize that commutativity and associativity are independent concepts and independent properties. They are easily (and often) confused. For some of these, the “usual” operations on $\mathbb{Z}$ might have the properties that you want. For some, you might have to make up some more unusual examples. Remember a binary operation on $\mathbb{Z}$ must apply to all pairs of elements in $\mathbb{Z}$ and must always produce an integer. So for example $x \star y = x^y$ does *not* define a binary operation on $\mathbb{Z}$ because (e.g.) 2^{-1} is not an integer.

4. **(1,2)** A binary operation $\star$ is defined on $\mathbb{Z}$ by $x \star y = |x - y|$. Determine, with explanation, whether $\mathbb{Z}$ is a group under the operation $\star$.

5. **(2,3,5)**$*$
   (a) Write out the multiplication table for the group D_8 of symmetries of the square.
   (b) Show that D_8 contains a cyclic subgroup of order 4.
   (c) Show that D_8 contains another subgroup of order 4 in which every element is its own inverse.

6. **(1,2,5,6)** $*$ Let G be a group with subgroups H and K.
   (a) Prove that $H \cap K$ must be a subgroup of G.
   (b) Give an example to show that $H \cup K$ is not necessarily a subgroup of G.

   Note: Your answer to part (a) should be a general proof that the set $H \cap K$ is closed under the operation of G, includes the identity element of G, and contains the inverse in G of each of its elements, provided that H and K are subgroups of G. Remember that what needs to be done to establish that some element x belongs to $H \cap K$ is to show that x belongs to H *and* x belongs to K. Your answer to part (b) should present a specific group with a pair of specific subgroups, whose union is not a group for some demonstrated reason. Examples of pairs of subgroups like this are plentiful, once you give yourself a particular group in which to work.

7. **(2,5,6)**$*$ Let D_6 be the group of symmetries of an equilateral triangle. Show that D_6 can be generated by one rotation and one reflection, or by two reflections.

8. Let D_8 be the group of symmetries of the square.
   (a) Show that D_8 can be generated by the rotation through 90° and any one of the four reflections.
   (b) Show that D_8 can be generated by two reflections.
   (c) Is it true that *any* choice of a pair of (distinct) reflections is a generating set of D_8?

   Note: What is mainly required here is patience. The first important step is to set up your notation in a clear way, so that you (and your reader) can see what you are doing. You might find it useful to write out the whole group table for D_8, which is a useful exercise anyway. Then for part (a), choose one of the four reflections, think about how it composes with the rotation through 90°, and how you can use this to obtain the remaining reflections. Try to explain why your argument would work for any of the four reflections. For parts (b) and (c), think about the geometry of the different pairs of reflections that you could choose. The composition of two reflections is always a rotation, but how does the angle of rotation depend on the two reflections that you choose?

9. $*$ Prove that the group $(\mathbb{Q}, +)$ of rational numbers under addition does not have a finite generating set.

   Note: This might be challenging. To think about it, take any finite set of rational numbers and think about which rational numbers you can get by adding them (and their negatives) together as much as you want. What you need to do is show that there are rational numbers that you definitely can’t get in this way. If you are in doubt, give yourself a particular collection of two or three specific rational numbers as your starting point and see why there are some elements of $\mathbb{Q}$ that you will definitely not be able to generate from those. You will need to use your knowledge about factorization of integers.

10. Prove that a subgroup of a cyclic group must be cyclic.

    Note: This is a well-known theorem in group theory that can be found in many of the standard texts. The key ingredients needed to prove it are the well-ordering of the integers and the division algorithm for the integers. This could be a topic for a poster (see the list of poster topics on the website if you might be interested in that).