Character Theory of Finite Groups 2005-2006

Problem Sheet 1

Due date for all Problems : Wednesday October 5

1. Let D_6 denote the dihedral group of order 6. By interpreting D_6 as a group of linear transformations of $\mathbb{R}^2$, construct a faithful irreducible matrix representation of D_6 of degree 2 over $\mathbb{R}$.

2. Suppose that $\rho : G \longrightarrow \mathrm{GL}(n, F)$ is a representation of a finite group G over a field F. Suppose that g and h are elements of G for which $\rho(g)\rho(h) = \rho(h)\rho(g)$. Does it follow that $gh = hg$ in G?

3. Let G be a group and let F be a field. Define a relation $\sim$ on the set of F-representations of G by declaring that $\rho_1 \sim \rho_2$ if ρ_2 is equivalent to ρ_1 in the sense of Section 1.2 of the lecture notes. Prove that $\sim$ is an equivalence relation.

4. Let F be a field. Prove that if A and B are $n \times n$ matrices with entries in F (for any positive integer n), then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

5. Let $\rho : G \longrightarrow \mathrm{GL}(n, \mathbb{R})$ be a representation of a group G over a field F. Prove that the function $\theta : G \longrightarrow F^{\times}$ defined by $\theta(g) = \det(\rho(g))$ for all $g \in G$ is a representation of G of degree 1.

6. Let G be a group of odd order, and let $\rho : G \longrightarrow \mathrm{GL}(n, \mathbb{R})$ be a representation of G over the field of real numbers. Prove that $\det(\rho(g)) = 1$ for all $g \in G$.

7. Let G be a finite group and let x and y be elements of G. Prove that the conjugacy classes $\mathrm{Cl}(x)$ and $\mathrm{Cl}(y)$ in G are either equal or disjoint.

8. Let G be a finite group and let $x \in G$. Prove that the centralizer $C_G(x)$ of x in G is a subgroup of G.

9. Let G be a finite group and let $x \in G$. If g and h are elements of G, prove that $g^{-1}xg = h^{-1}xh$ of and only if hg^{-1} belongs to the centralizer $C_G(x)$ of x in G. Deduce that $|\mathrm{Cl}(x)| = [G : C_G(x)]$.

10. By considering D_8 as a group of permutations of the vertices of a square, construct a faithful permutation representation of degree four of D_8. Describe the character of this representation and write it as a sum of the irreducible complex characters of D_8 as described in Section 1.4.

Comments and Hints overleaf

Remarks on the Problems

1. This can be done exactly as we constructed our irreducible representation of D_8 of degree 2 in Section 1.1. If you consider D_6 to be the group of symmetries of an equilateral triangle with vertices at $(0,1), (-\sqrt{3}/2, -1/2)$ and $(\sqrt{3}/2, 1/2)$, each of these symmetries is a linear transformation of $\mathbb{R}^2$. The elements of D_6 can be written as $\mathrm{id}, x, x^2, y, xy, x^2y$, where x is the rotation through $2\pi/3$ counterclockwise about the origin, and y is the reflection in the X-axis.

2. This is really a question about group homomorphisms.

3. Recall that an equivalence relation is one that is reflexive ($x \sim x$ for all x), symmetric ($x \sim y$ implies $y \sim x$) and transitive ($x \sim y$ and $y \sim z$ implies $x \sim z$).

4. This involves writing the entries on the main diagonal of AB in terms of the entries of A and B.

5. This involves a well-known property of determinants, which you should state (without proof).

6. This is related to Question 5 above. Remember that in a finite group every element has finite order, and that the order of each element divides the order of the group.

10. Suppose the vertices of the square are labelled (in counterclockwise order) A, B, C and D. Then the rotation through $\pi/2$ can be interpreted as the permutation of $\{A, B, C, D\}$ that sends A to B, B to C, C to D and D to A. The reflection in the diagonal AC is the permutation that swaps B and D and leaves A and C fixed.