MA500-1 ADVANCED LINEAR ALGEBRA
Problem Sheet 1
Due date Wednesday October 11

Let A be a $m \times n$ matrix with entries in a field $\mathbb{F}$. Recall that the *row rank* of a A is the dimension of the subspace of $\mathbb{F}^n$ spanned by its rows, and the *column rank* of A is the dimension of the subspace of $\mathbb{F}^m$ spanned by its columns.

1. Below are three "proofs" that the row rank and column rank are always the same.

   A By induction on n, the number of columns. Let A be a $m \times n$ matrix and suppose that the statement is true for matrices with fewer than n columns. If the first column of A is full of zeros, then the row and column rank of A are equal to the row and column rank of the matrix obtained from A by deleting Column 1. These are equal to each other by the induction hypothesis.

   If the first Column of A is not full of zeros, then we may apply elementary row operations to A to obtain a new matrix A_1 whose first column is $(1\ 0\ \cdots\ 0)^T$. Let A_1' be the matrix obtained from A_1 by deleting the first row and first column. It is clear that the row and column ranks of A_1 exceed those of A_1' by 1. Since the induction hypothesis applies to A_1' its row and column ranks coincide, thus so do the row and column ranks of A_1 and hence A.

   B The row rank of A is the column rank of A^T, the transpose of A. Since the column rank of a matrix and its transpose are equal, it follows that the row rank of A is equal to the column rank of A^T and hence to the column rank of A.

   C Let c and r denote the column and row ranks respectively of A, and let C be a $m \times c$ matrix whose columns span the column space of A. Since every column of A is a linear combination of the columns of C, this means that $A = CB$ for some $c \times n$ matrix B. But now every row of A is a linear combination of the c rows of B, which means that the row rank of A is at most c. Hence $r \leqslant c$.

   A similar argument expresses A as the product DR, where R is a $r \times n$ matrix whose rows form a basis for the rowspace of A, and D is some $m \times r$ matrix. Since every column of A is a linear combination of the r columns of D it follows that $c \leqslant r$.

   Since $r \leqslant c$ and $c \leqslant r$ we conclude that $r = c$.

   For each of the three proposed proofs, answer the following question.

   If you were assessing this proof for inclusion in a textbook for a course like this, what would you recommend to the editor?

   Your answer should include the following elements.

   - Whether the proof is basically correct or not;
   - If it is correct, whether it is explained in sufficient detail and how it should be amended if not;
   - If it is wrong, why it is wrong;
   - Whether you would advise someone to read it if they want to understand why the row rank and column ranks are equal.
   - Any comments on whether you prefer some of these proofs to others and why.

further problems overleaf

The *rank* of a matrix is the dimension of its row space or its column space; since these are the same there is no ambiguity in referring to *rank* rather than *row rank* or *column rank*. The next problem concerns the Rank-Nullity Theorem.

2. (a) Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. The *kernel* of T is the subspace of V consisting of all elements v for which $T(v) = 0$. The *image* of T is the subspace of W consisting of all elements of the form $T(v)$ where $v \in V$. Prove that

$$\dim(\ker T) + \dim(\mathrm{Im} T) = \dim V.$$

   (b) Let A be a $m \times n$ matrix with entries in a field $\mathbb{F}$. The *right nullspace* N_R of A is the subspace of $\mathbb{F}^n$ consisting of all vectors v for which $Av = 0_{m\times 1}$. Use part (a) to prove that

$$\dim(N_R) + \mathrm{rank}(A) = n.$$

   (c) Define the *left nullspace* of a matrix A and write down the version of the statement of part (b) above that involves the left nullspace.

3. Suppose that P is an invertible matrix in $M_n(\mathbb{F})$ and that A is a matrix with n rows. Prove that $\mathrm{rank}(PA) = \mathrm{rank}(A)$.
   If $\mathrm{rank}(PA) = \mathrm{rank}(A)$, must it follow that P is invertible?

4. For $p \times n$ matrices A and B, prove that

$$|\mathrm{rank}(A) - \mathrm{rank}(B)| \leqslant \mathrm{rank}(A + B) \leqslant \mathrm{rank}(A) + \mathrm{rank}(B).$$

5. For matrices $A \in M_{p\times q}(\mathbb{F})$ and $B \in M_{q\times n}(\mathbb{F})$, prove that

$$\mathrm{rank}(A) + \mathrm{rank}(B) - q \leqslant \mathrm{rank}(AB) \leqslant \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

Comments overleaf

Remarks on the problems

Many of these problems are challenging, but you have a few weeks to think about them and discuss them, which will hopefully be enough time to get used to the kind of thinking involved. This page has a few general comments and some ideas that might help you to get started. Maybe it's good to have a go at the problems first without reading the comments though. For some of them there are lots of good approaches that don't necessarily follow the lines suggested in these remarks.

1. Your task here is to study each of the proposed proofs and assess whether it is correct, whether it does what it claims, whether it is explained clearly and sufficiently, whether it is convincing. If you think that a proof is not correct as it stands, consider whether it can be corrected or not. It is not uncommon for a statement to have more than several correct proofs involving different ideas, and preferring one to another is to some extent a matter of taste.

2. This is a standard and important theorem of linear algebra. One approach to proving it is to start with a basis of $\ker T$, extend it to a basis of V by adding more elements, look at the images of these added elements in the W and show that they form a basis for $\mathrm{Im}T$.

3. Note that every row of PA is a linear combination of the rows of A. What does this tell you about the relationship between the ranks of PA and A? Now what about $P^{-1}(PA)$?

4. There are two parts here. The easier one (I think) is to show that $\mathrm{rank}(A+B) \leqslant \mathrm{rank}(A)+\mathrm{rank}(B)$. To do this, think about the rows of $A+B$. Each of them is the sum of a row of A and a row of B. So you can use a basis for the rowspace of A and a basis for the rowspace of B to get an upper bound for the dimension of the rowspace of $A+B$.
   For the other inequality, first assume that $\mathrm{rank}(A) \geqslant \mathrm{rank}(B)$. Then you need to prove that $\mathrm{rank}(A) - \mathrm{rank}(B) \leqslant \mathrm{rank}(A+B)$. Can you rewrite this in order to obtain it as a consequence of the first part of the problem?

5. Again there are two things to do here. The easier one is to show that $\mathrm{rank}(AB) \leqslant \mathrm{rank}(A)$ and $\mathrm{rank}(AB) \leqslant \mathrm{rank}(B)$.
   For the other inequality, think of B as a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^q$, and of A as a linear transformation from $\mathbb{F}^q$ to $\mathbb{F}^p$. Then AB represents the composition of the two, which goes from $\mathbb{F}^n$ to $\mathbb{F}^p$. You need to figure out the minimum possible dimension of the image of this mapping. Bear in mind that the dimension of the kernel of the transformation described by A is $q-\mathrm{rank}(A)$, how this intersects the image of B is a key consideration.