1 Linear transformations and eigenvalues

In this course we will be thinking about linear algebra and about matrix theory. These subjects are closely related and sometimes are probably considered to be indistinguishable. From an algebraic viewpoint, maybe they could be described as follows.
Linear Algebra is the area of abstract algebra that is concerned with vector spaces and mappings between them that respect their algebraic structure, which are linear transformaions. In the same way, group theory deals with groups and group homomorphisms, and so on. Single variable calculus deals with subsets of $\mathbb{R}$ and not with all functions between them but with those that are amenable to the machinery of calculus such as continuous or differentiable functions (or whatever).
Matrix Theory is the study of the arithmetic of matrices including basis algebraic operations (addition, multiplicaton, inversion) as well as topics such as factorization, properties of eigenvalues and eigenvectors, special features (for example symmetry, positivity, positive-definiteness, diagonalizabilty, canonical forms, the list goes on . . .). The subject includes existence theorems about special forms, properties etc as well as computational methods (and their scope and limitations).

Becasue a matrix can be considered to represent a linear transformation after a choice of basis, and because composition of linear transformations in this context corresponds to matrix multiplication, more or less every question in linear algebra can be translated to a question about matrices. The reverse is true also although the issue of choosing a (special) basis is always present in both directions of this translation. The upshot is that abstract linear algebra has a relatively concrete manifestation in the world of matrices, which makes it conducive not only to building operational understanding but also to computation. This is probably why linear algebra is considered (at least by pedagogists) to be more accessible and "concrete" than other areas of abstract algebra.

Anyway, on to some subject matter.

Definition 1.1. *Let V be a vector space over a field $\mathbb{F}$. A* linear transformation *of V is a function $T : V \to V$ that has the following properties:*

- $T(u + v) = T(u) + T(v)$ *for all* $u, v \in V$.
- $T(kv) = kT(v)$ *for all* $v \in V$ *and for all* $k \in \mathbb{F}$.

So T preserves the operations of addition and multiplication by scalars.
Note: If you don't like dealing with an "arbitary field $\mathbb{F}$", just think of $\mathbb{F}$ as being the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Thinking about examples of linear transformations of $\mathbb{R}^2$ or $\mathbb{R}^3$ can give us a very good geometric sense which is still valuable even when the geometry can't be so easily visualized, for example if we are in higher dimensions or working over some more obscure field.

Example 1.2. *Let $\theta \in [0, 2\pi)$. The rotation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotates every vector v (visualized as an arrow pointing from the origin) through the angle θ in a counter-clockwise sense.*

From the geometric definition of addition of vectors (which says that the "terminal point" of the vector $u + v$ is the fourth vertex of the parallelogram that has the origin and the "terminal points" of u and v as three of its vertices) it is clear that T_θ preserves addition. It is also clear from the definition that T_θ respects multiplication by scalars.

A *line* in a vector space is a one-dimensional subspace, consisting of all scalar multiples of a particular vector. Every non-zero vector belongs to exactly one line, and two non-zero vectors belong to the same line if and only if they are scalar multiples of each other. The zero vector belongs to every line.

The image of a line under a linear transformation will always be a line, since if two vectors are scalar multiples of each other, then so are their images under any linear transformation.
Exercise: Is it true that every function from a vector space to itself that maps lines to lines is a linear transformation?

Definition 1.3. *Let $T : V \to V$ be a linear transformation. A non-zero vector $v \in V$ is called an* eigenvector *of T if $T(v) = \lambda v$ for some scalar v, called the* eigenvalue *of T to which v corresponds.*

So eigenvectors correspond to lines that are preserved (or mapped to themselves by T). From the geometry of $\mathbb{R}^2$ we can observe that, provided that $\theta \neq 0$ and $\theta \neq \pi$, the rotation T_θ has no eigenvector in $\mathbb{R}^2$ - it moves every line to a different line. So it is not necessarily always true that a linear transformation of a vector space V will have an eigenvector in V.

Every non-zero vector $v \in \mathbb{R}^2$ is an eigenvector of T_π corresponding to the eigenvalue -1, and every non-zero vector is an eigenvector of T_0 corresponding to 1 (not so surprising since T_0 is the identity function on $\mathbb{R}^2$).