24 Positive matrices - the Frobenius-Perron Theorem

In this section we are considering matrices whose entries belong to the field $\mathbb{R}$ of real numbers. In $\mathbb{R}$ (but not in other fields such as $\mathbb{C}$ for example), non-zero elements are either positive or negative, in fact $\mathbb{R}$ is an example of an ordered field. This is a very familiar property but actually it is quite special. It allows us to define the notion of a *positive matrix* and to investigate what special properties positive matrices might have.

Definition 24.1. *A matrix in* $M_{p\times q}(\mathbb{R})$ *is* positive *if all of its entries are positive (and* non-negative *if all of its entries are non-negative). We write* $A > 0$ *and* $A \geqslant 0$ *to indicate that a matrix* A *is positive or non-negative. The difference is that some entries of a non-negative matrix may be zero.*

Positive square matrices (and certain classes of non-negative matrices) have some special spectral properties that are often collected together into the statement of the Frobenius-Perron Theorem.

Theorem 24.2. *Let* A *be a* $n \times n$ *positive matrix with spectral radius* ρ. *Then* $\rho > 0$ *and*

1. ρ *is an eigenvalue of* A.
2. ρ *has algebraic multiplicity 1 as an eigenvalue of* A.
3. *There is an eigenvector* v *of* A *corresponding to* ρ *that has all of its entries positive.*
4. *If* λ *is an eigenvalue of* A *and* $\lambda \neq \rho$, *then* $|\lambda| < \rho$.
5. *If* u *is an eigenvector of* A *(corresponding to any eigenvalue) whose entries are all positive, then* u *is a scalar multiple of* v *(from 3. above).*

The key point of Theorem 24.2 is Item 1, which is more significant than it might look at first glance. In general, for a matrix A in $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$, the spectral radius is just the maximum of the moduli of the eigenvalues. In general there is no reason to expect that the spectral radius is itself an eigenvalue, since the eigenvalue of greatest modulus need not be real, and if it is real it need not be positive. So the situation for positive matrices is really special - there is an eigenvalue of greatest modulus that is a positive real number (this is sometimes called the *Perron root*. Not only that but every other eigenvalue has modulus strictly less than that of the Perron root, there is only one eigenvalue on the circle of radius ρ. Not only that, but this eigenvalue has a corresponding eigenvector in which all entries are positive, and no other eigenvalue has this property.

Example 24.3. *Let* $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$.
The characteristic polynomial of A *is* $x^2 - 6x - 7$ *or* $(x-7)(x+1)$; *the eigenvalues are* 7 *and* -1. *The spectral radius is 7. Eigenvectors corresponding to* $\lambda = 7$ *and* $\lambda = 1$ *respectively are* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and* $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Before proving this theorem in general, we consider what it says about the 2×2 case. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with positive real entries. Let the eigenvalues of A be λ and μ. Then

- $\mu + \lambda = a + d$.
- $\mu\lambda = ad - bc$

Either μ and λ are both real or they are complex numbers that are complex conjugates of each other. We first show that they must be real.

Lemma 24.4. $\mu \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}$.

Proof. Suppose not, and write $\mu = x + yi$, $\lambda = x - iy$. Then $2x = a + d$ and $x^2 + y^2 = ad - bc$. Now

$$\begin{aligned} x^2 + y^2 &= \frac{1}{4}(a^2 + d^2 + 2ad) + y^2 = ad - bc \\ &\Longrightarrow \frac{1}{4}(a - d)^2 + y^2 = -bc \end{aligned}$$

Since $-bc$ is negative, this is impossible. □

Thus the eigenvalues of A are both real, and since their sum is positive at least one of them is positive. If one is positive and one negative, then the positive one must have the greater absolute value. Note that A cannot have a repeated real eigenvalue, thus is ruled out by the above argument with $y = 0$. This proves items 1, 2 and 4 of the Theorem for the 2×2 case.

Now suppose that $\mu < \lambda$. Then

$$\begin{aligned} \lambda + \mu &= a + d \\ \Longrightarrow 2\lambda &> a + d \\ \Longrightarrow \lambda &> \frac{a + d}{2}. \end{aligned}$$

This means that either $\lambda > a$ or $\lambda > d$ (or both). Let $v \in \mathbb{R}^2$ be an eigenvector of A, with $v = \binom{v_1}{v_2}$. Then neither v_1 nor v_2 can be equal to zero. To see this note that if $v_1 = 0$ and $v_2 \neq 0$, then the first component of Av is not zero. Thus we can choose v with $v_1 = 1$. Then

$$Av = \lambda v \Longrightarrow a + bv_2 = \lambda, \;\; c + dv_2 = \lambda v_2.$$

If $\lambda > a$ then since b is positive it follows from the first equation that v_2 is positive. If $\lambda \not> a$ then $\lambda > d$ and the second equation says $c = (\lambda - d)v_2$. Since c and $\lambda - d$ are positive, it follows that v_2 is positive. Thus both entries of v are positive, which proves part 3. of the theorem in the 2×2 case.

Finally, let $u = \binom{u_1}{u_2}$ be an eigenvector of A corresponding to μ, and as above suppose that $u_1 = 1$. As above we have

$$\mu + \lambda = a + d \Longrightarrow 2\mu < a + d \Longrightarrow \mu < a \text{ or } \mu \leqslant d$$

and

$$Au = \mu u \Longrightarrow a + bu_2 = \mu \text{ and } c + du_2 = \mu u_2.$$

If $\mu < a$ then the first equation implies that $u_2 < 0$. If $\mu \not< a$ then $\mu < d$ and the second equation says $c = (\mu - d)u_2$ which means that u_2 is negative since c is positive. So the entries of u are of opposite sign. This completes the proof of the Frobenius-Perron Theorem for $n = 2$.