Chapter 2

Real Symmetric Matrices

2.1 Special properties of real symmetric matrices

A matrix $A \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$) is *diagonalizable* if it is similar to a diagonal matrix. If this happens, it means that there is a basis of $\mathbb{C}^n$ with respect to which the linear transformation of $\mathbb{C}^n$ defined by left multiplication by A has a diagonal matrix. Every element of such a basis is simply multiplied by a scalar when it is multiplied by A, which means exactly that the basis consists of eigenvectors of A.

Lemma 2.1.1. *A matrix $A \in M_n(\mathbb{C})$ is diagonalizable if and only if $\mathbb{C}^n$ possesses a basis consisting of eigenvectors of A.*

Not all matrices are diagonalizable. For example $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not. To see this note that 1 (occurring twice) is the only eigenvalue of A, but that all eigenvectors of A are scalar multiples of $\binom{1}{0}$, so $\mathbb{C}^2$ (or $\mathbb{R}^2$) does not contain a basis consisting of eigenvectors of A, and A is not similar to a diagonal matrix.

We note that a matrix can fail to be diagonalizable only if it has repeated eigenvalues, as the following lemma shows.

Lemma 2.1.2. *Let $A \in M_n(\mathbb{R})$ and suppose that A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$. Then A is diagonalizable.*

Proof. Let v_i be an eigenvector of A corresponding to λ_i. We will show that $S = \{v_1, \dots, v_n\}$ is a linearly independent set. Since v_1 is not the zero vector, we know that $\{v_1\}$ is linearly independent. If S is linearly dependent, let k be the least for which $\{v_1, \dots, v_k\}$ is linearly dependent. This means that $\{v_1, \dots, v_{k-1}\}$ is a linearly independent set and

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$$

for some $a_i \in \mathbb{C}$, not all zero. Multiplying this equation on the left separately by A and by λk gives

$$\begin{aligned} \lambda_k v_k &= a_1\lambda_1 v_1 + a_2\lambda_2 v_2 + a_{k-1}\lambda_{k-1} v_{k-1} \\ a_1\lambda_k v_1 + a_2\lambda_k v_2 + a_{k-1}\lambda_k v_{k-1} & \\ \Longrightarrow 0 &= a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_2 - \lambda_k) + \cdots + a_k(\lambda_{k-1} - \lambda_k)v_{k-1}. \end{aligned}$$

Since the complex numbers $\lambda_i - \lambda_k$ are non-zero for $i = 1, \dots, k-1$ and at least one of these a_i is non-zero, the above is an expression for the zero vector as a nontrivial linear combination of $v_1, \dots, v_{k-1}$, contrary to the choice of k. We conclude that S is linearly independent and hence that it is a basis of $\mathbb{C}^n$. □

Definition 2.1.3. *A matrix $A \in M_n(\mathbb{R})$ is* symmetric *if it is equal to its transpose, i.e. if $A_{ij} = A_{ji}$ for all i and j.*

Symmetric matrices arise naturally in various contexts, including as adjacency matrices of undirected graphs. Fortunately they have lots of nice properties. To explore some of these we need a slightly more general concept, that of a complex Hermitian matrix.

Definition 2.1.4. *Let $A \in M_n(\mathbb{C})$. The* Hermitian transpose, *or* conjugate transpose *of A is the matrix A^* obtained by taking the transpose of A and then taking the complex conjugate of each entry. The matrix A is said to be* Hermitian *if $A = A^*$.*

Notes

1. *Example*: If $A = \begin{pmatrix} 2+i & 4-i \\ 3 & 3-i \end{pmatrix}$, then $A^* = \begin{pmatrix} 2-i & 3 \\ 4+i & 3+i \end{pmatrix}$

2. The Hermitian transpose of A is equal to its (ordinary) transpose if and only if $A \in M_n(\mathbb{R})$. In some contexts the Hermitian transpose is the appropriate analogue in $\mathbb{C}$ of the concept of transpose of a real matrix.

3. If $A \in M_n(\mathbb{C})$, then the trace of the product A^*A is the sum of all the entries of A, each multiplied by its own complex conjugate (check this). This is a non-negative real number and it is zero only if $A = 0$. In particular, if $A \in M_n(\mathbb{R})$, then trace(A^TA) is the sum of the squares of all the entries of A.

4. Suppose that A and B are two matrices for which the product AB exists. Then $(AB)^T = B^TA^T$ and $(AB)^* = B^*A^*$ (it is routine but worthwile to prove these statements). In particular, if A is any matrix at all, then
$$(A^TA)^T = A^T(A^T)^T = A^TA, \text{ and } (A^*A)^* = A^*(A^*)^* = A^*A,$$
so A^TA and A^*A are respectively symmetric and Hermitian (so are AA^T and AA^*).

The following theorem is the start of the story of what makes real symmetric matrices so special.

Theorem 2.1.5. *The eigenvalues of a real symmetric matrix are all real.*

Proof. We will prove the stronger statement that the eigenvalues of a complex Hermitian matrix are all real. Let A be a Hermitian matrix in $M_n(\mathbb{C})$ and let λ be an eigenvalue of A with corresponding eigenvector v. So $\lambda \in \mathbb{C}$ and v is a non-zero vector in $\mathbb{C}^n$. Look at the product v^*Av. This is a complex number.
$$v^*Av = v^*\lambda v = \lambda v^*v.$$
The expression v^*v is a positive real number, since it is the sum of the expressions $\bar{v_i}v_i$ over all entries v_i of v.

We have not yet used the fact that $A^* = A$.
Now look at the Hermitian transpose of the matrix product v^*Av.
$$(v^*Av)^* = v^*A^*(v^*)^* = v^*Av.$$
This is saying that v^*Av is a complex number that is equal to its own Hermitian transpose, i.e. equal to its own complex conjugate. This means exactly that $v^*Av \in \mathbb{R}$.

We also know that $v^*Av = \lambda v^*v$, and since v^*v is a non-zero real number, this means that $\lambda \in \mathbb{R}$. □

So the eigenvalues of a real symmetric matrix are real numbers. This means in particular that the eigenvalues of the adjacency matrix of an undirected graph are real numbers, they can be arranged in order and we can ask questions about (for example) the greatest eigenvalue, the least eigenvalue, etc.

Another concept that is often mentioned in connection with real symmetric matrices is that of positive definiteness. We mentioned above that if $A \in M_{m\times n}(\mathbb{R})$, then $A^T A$ is a symmetric matrix. However not every symmetric matrix has the form $A^T A$, since for example the entries on the main diagonal of $A^T A$ do not. It turns out that those symmetric matrices that have the form $A^T A$ (even for a non-square A) can be characterized in another way.

Definition 2.1.6. *Let A be a symmetric matrix in $M_n(\mathbb{R})$. Then A is called* positive semidefinite (PSD) *if $v^T A v \geqslant 0$ for all $v \in \mathbb{R}^n$. In addition, if $v^T A v$ is strictly positive whenever $v \neq 0$, then A is called* positive definite *(PD).*

Notes

1. The identity matrix I_n is the classical example of a positive definite symmetric matrix, since for any $v \in \mathbb{R}^n$, $v^T I_n v = v^T v = v \cdot v \geqslant 0$, and $v \cdot v = 0$ only if v is the zero vector.

2. The matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is an example of a matrix that is *not* positive semidefinite, since

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -2.$$

   So positive (semi)definite is not the same thing as positive - a symmetric matrix can have all of its entries positive and still fail to be positive (semi)definite.

3. A symmetric matrix can have negative entries and still be positive definite, for example the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ is SPD (symmetric positive definite). To see this observe that for real numbers a and b we have

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a^2 - ab - ab + 2b^2 = (a-b)^2 + b^2.$$

   Since $(a-b)^2 + b^2$ cannot be negative and is zero only if both a and b are equal to zero, the matrix A is positive definite.

The importance of the concept of positive definiteness is not really obvious at first glance, it takes a little bit of discussion. We will defer this discussion for now, and mention two observations related to positive (semi)definiteness that have a connection to spectral graph theory.

Lemma 2.1.7. *The eigenvalues of a real symmetric positive semidefinite matrix are non-negative (positive if positive definite).*

Proof. Let λ be an eigenvalue of the real symmetric positive semidefinite matrix A, and let $v \in \mathbb{R}^n$ be a corresponding eigenvector. Then

$$0 \leqslant v^T A v = v^T \lambda v = \lambda v^T v.$$

Thus λ is nonnegative since $v^T v$ is a positive real number. □

Lemma 2.1.8. *Let $B \in M_{n\times m}(\mathbb{R})$ for some positive integers m and n. Then the symmetric matrix $A = BB^T$ in $M_n(\mathbb{R})$ is positive semidefinite.*

Proof. Let $u \in \mathbb{R}^n$. Then

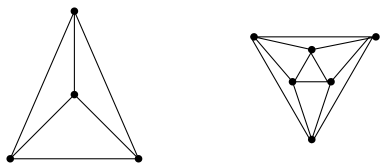
$$u^T A u = u^T B B^T u = (u^T B)(B^T u) = (B^T u)^T (B^T u) = (B^T u) \cdot (B^T u) \geqslant 0,$$

so A is positive semidefinite. □

The next section will contain a more detailed discussion of positive (semi)definiteness, including the converses of the two statements above. First we digress to look at an application of what we know so far to spectral graph theory.

Definition 2.1.9. *Let* G *be a graph. The* line graph *of* G*, denoted by* $L(G)$*, has a vertex for every edge of* G*, and two vertices of* $L(G)$ *are adjacent if and only if their corresponding edges in* G *share an incident vertex.*

Example 2.1.10. K_4 *(left) and its line graph (right).*



Choose an edge of K_4. Since each of its incident vertices has degree 3, there are four other edges with which it shares a vertex. So the vertex that represents it in $L(K_4)$ has degree 4. In general, if a graph G is regular of degree k, then $L(G)$ will be regular of degree $2k-2$. For any graph G, a vertex of degree d in G corresponds to a copy of the complete graph K_d within $L(G)$.

Not every graph can be a line graph. For example, a vertex of degree 3 in a line graph $L(G)$ must have the property that at least two of its neighbours are adjacent to each other, because it corresponds to an edge e of the graph G that shares a vertex with three other edges. At least two of these three must be incident with the same vertex of e. Thus (for example) $L(G)$ can be a tree or forest only if $L(G)$ has no vertex of degree exceeding 2, which means that G is a collection of disjoint paths (in this case $L(G)$ is also a collection of disjoint paths, of lengths one less than the paths of G itself). The cycle C_n is its own line graph. The line graph of the path P_n is P_{n-1}. The line graph of the star on n vertices (which has one vertex of degree $n-1$ and $n-1$ of degree 1) is the complete graph K_n.

Line graphs all share the following spectral property, which is remarkable easy to prove using what we already know about positive semidefinite matrices.

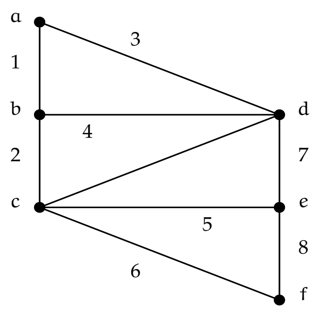
Theorem 2.1.11. *Let* $L(G)$ *be the line graph of a graph* G*, and let* $A(L(G))$ *be the adjacency matrix of* $L(G)$*. Then every eigenvalue of* $L(G)$ *is at least equal to* -2*.*

To prove Theorem 2.1.11 we need one more device that links matrices to graphs.

Definition 2.1.12. *Let* G *be a graph with* n *vertices and* m *edges. The* incidence matrix *of* G*, denoted* $B(G)$*, is the* $m \times n$ $(0,1)$*-matrix with rows indexed by the edges of* G *and columns by the vertices of* G*, that has a 1 in the* (i,j)*-position if and only if the edge labelling Row i is incident with the vertex labelling Column* j*.*

The incidence matrix depends on a choice of ordering of both the vertices and the edges obviously.

Example 2.1.13. *A graph and its incidence matrix.*



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Each row of an incidence matrix has two 1s, and the number of 1s in a column is the degree of the corresponding vertex.

Now suppose that B is the incidence matrix of a graph G, and consider the positive semidefinite matrices BB^T and B^TB.

The rows and columns of BB^T are indexed by the edges of G. The entry in the (i, j) position of BB^T is the scalar product of Rows i and j of B, each of which has exactly two entries equal to 1. If $i = j$ then the entry in the (i, i) position of BB^T is 2. If $i \neq j$, then Rows i and j of B are different since they represent different edges e_i and e_j respectively of G. In this case $(BB^T)_{ij}$ is equal to 1 if the edges e_i and e_j have a vertex in common, and 0 otherwise. Thus an off-diagonal entries of BB^T is 1 or 0 according as the edges of G labelling its row and column share an incident vertex or not. The diagonal entries are all 2. Thus $BB^T - 2I$ is exactly the adjacency matrix of the line graph of G, or

$$BB^T = 2I + A(L(G)).$$

Theorem 2.1.14. *Let* $L(G)$ *be the line graph of a graph* G*, and let* λ *be the least eigenvalue of the adjacency matrix of* $L(G)$*. The* $\lambda \geqslant -2$*.*

Proof. From the above description of BB^T we know that $2I + A(L(G))$ is a positive semidefinite matrix and so its eigenvalues are all non-negative. Moreover the spectrum of $2I + A(L(G))$ are obtained by adding 2 to each item in the spectrum of $A(L(G))$, so $\lambda + 2 \geqslant 0 \Longrightarrow \lambda \geqslant -2$. □

It is not true unfortunately that a graph must be a line graph if all eigenvalues of its adjacency matrix are at least -2.

Now we turn to the matrix B^TB. The rows and columns of this matrix are labelled by the vertices of G and the entry in the (i, j) positive is the scalar product of Column i and Column j of B. If $i = j$, this is the degree of Vertex i. If $i \neq j$, then Column i and Column j have a 1 in the same position if and only if Vertex i and $Vertex j$ belong to the same edge. This happens in exactly one position if the vertices i and j are adjacent in G, and in no position if they are non-adjacent. Thus the entry in the off-diagonal position (i, j) of B^TB is 1 if Vertices i and j are adjacent in G and 0 otherwise. This means that away from the main diagonal, B^TB coincides with the adjacency matrix of G. Putting all of this together gives

$$B^TB = \Delta + A(G),$$

where $A(G)$ is the adjacency matrix of G and $Delta$ is the diagonal matrix whose entry in the (i, i)-position is the degree of vertex v_i. We have shown the the matrix $\Delta + A(G)$ is positive semidefinite for every graph G. In the special case where G is regular of degree k, this shows that every eigenvalue of G is at least $-k$.