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Problems about Rank

- Given a partial matrix, what is the range of ranks of its completions?
- Characterize (all, or extremal examples of) partial matrices whose completions satisfy specified rank bounds, e.g. have constant rank.
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**Theorem (adapted from Huang and Zhan (2011))**

Let $A$ be a $m \times n$ partial matrix of constant rank $r$ over a field $\mathbb{F}$. If $|\mathbb{F}| \geq \max(m, n)$ then $A$ possesses a $r \times r$ sub(partial)matrix whose completions all have rank $r$. 
The following $3 \times 4$ partial matrix over $\mathbb{F}_2$ has all completions of rank 3, but possesses no $3 \times 3$ submatrix of constant rank 3.

\[
\begin{pmatrix}
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1 & 1 & Y & 0 \\
1 & 0 & 1 & Z
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Some Observations

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- $A$ possesses constant columns (assumed linearly independent).
- Let $C$ be the subspace of $\mathbb{F}^m$ spanned by the constant columns. Then $1 \leq \dim C \leq r - 2$ and every element of $C^\perp$ includes at least one zero entry.
- If $|\mathbb{F}| \geq r$, then $\dim C \leq |\mathbb{F}| - 2$, and $C$ includes an element with exactly one non-zero entry. An induction argument produces an $r \times r$ submatrix of $A$ of constant rank $r$. 

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Exceptional cases occur only if $|F| < r$

The following theorem can be proved by induction on $r$.

**Theorem**

There exist exceptional $m \times n$ (with $m \leq n$) partial matrices of constant rank $r$ over $\mathbb{F}_q$ if and only if $r > q$ and $n \geq r + q - 1$.

The base case: $r = q + 1, n \geq 2q$

An example with $q = 3 : (q + 1) \times (2q)$, exceptional of constant rank 4.

$$
\begin{pmatrix}
1 & 1 & X & 1 & 1 & 1 \\
1 & 2 & 1 & Y & 1 & 1 \\
2 & 0 & 1 & 1 & Z & 1 \\
0 & 2 & 2 & 1 & 1 & W
\end{pmatrix}
$$
The case \( r = q + 1 \): need at least \( 2q \) columns

Let \( A \) be a partial \( m \times n \) matrix over \( \mathbb{F}_q \) (\( m \leq n \)) of constant rank \( q + 1 \), and let \( C \subset \mathbb{F}_q^m \) be the span of the constant columns of \( A \).

- If \( \dim C \geq q \), then \( A \) is not exceptional.
- If \( C \) contains an element with exactly one non-zero entry, then \( A \) has a \( (m - 1) \times (n - 1) \) submatrix of constant rank \( q \), and \( A \) is not exceptional.
- Otherwise \( C^\perp \) has the “distributed zero property”: every element of \( C^\perp \) has at least one zero entry, but there is no position that is always zero in \( C^\perp \).
- This means \( \dim C \geq q - 1 \), so if \( A \) is exceptional, \( \dim C = q - 1 \) and \( A \) has (exactly) \( q - 1 \) constant columns.
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The case $r = q + 1$: at least $q + 1$ indeterminate columns

A is a partial $m \times n$ matrix over $\mathbb{F}_q$ ($m \leq n$) of constant rank $q + 1$, and $C^\perp$ has the distributed zero property.

- Form $A'$ by assigning a value to all but one indeterminate in each indeterminate column of $A$.

- Given any $q$ positions in $\mathbb{F}_q^m$, there is an element $v$ of $C^\perp$ that has non-zero entries in all of them (this is because a vector space over $\mathbb{F}_q$ cannot be the union of $q$ hyperplanes).

- The indeterminates of $A'$ must collectively occupy at least $q + 1$ rows, otherwise $A'$ would have completions of different ranks.

- So $A'$ has at least $q + 1$ indeterminate columns, hence at least $2q$ columns in all.
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THANK YOU!

Advertisement If you are interested in this, see the talk by James McTigue on Thursday.